LECTURE 13

FUNCTIONS OF TWO VARIABLES

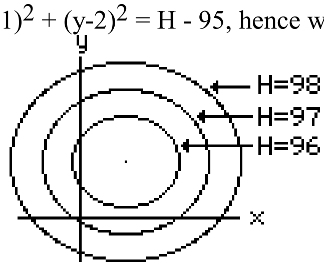
To date we've been studying calculus-in-the-plane, meaning all our functions ($y = f(x)$ or $x = f(t)$, $y = g(t)$) can be represented by curves lying in a **two** dimensional x-y plane and points are identified by **two** numbers (x, y) or (r,θ). But it's often the case (perhaps more often than not) that functions which occur in real life (meaning outside of calculus courses) depend upon more than a single variable.

- The temperature T at a point depends upon the location of the point and this location will, in general, be given by three coordinates (x,y,z) in some 3-dimensional space (maybe x = latitude, y = longitude, z = elevation): $T = f(x,y,z)$. Perhaps the temperature changes with time as well, so $T = f(x,y,z,t)$ where (x,y,z) gives the location of the point and t is the time when T is measured.
- The speed v of the water molecules in a stream depends upon where in the stream the speed is measured ... and perhaps the time when the measurement is taken: $v = f(x,y,z,t)$.
- In manufacturing, the cost per item may depend upon how many are manufactured: $C = f(N)$, meaning that it will cost \$C per item, if N items are manufactured. However, the cost may also depend upon the time of year because of scarcity of materials or labour, so we'd have $C = f(N,t)$ where t gives the time. (For example, t = 17 may mean 17 weeks from January 1.)
- The growth of a plant, H (measured perhaps in *cm/day*), may depend upon I the intensity of light provided (in *candlepower*), h the amount of humus in the soil (g/cm^3) and N the daily amount of nitrogen supplied (*g/day*): $H = f(I,h,N)$.

To address this problem ... a calculus for many dimensions ... we first begin with a function of just two variables which we write $z = f(x,y)$. Since a picture is worth a thousand words we need a way to picture the behaviour of this relationship, and to make things more concrete we consider a particular real-world problem.

LEVEL CURVES

Suppose x and y were distances (in kilometres) east-west and north-south from some fixed point. That is, x = 5, y = -7 means we're 5 km east and 7 km south of the fixed point. Suppose, too, that the elevation (above sea level) at (x,y) was given by: $z = x^2 + y^2 - 2x - 4y + 100$ metres. We could get some idea of the elevation at various points (x,y) by asking "Where are all the points with elevation = 96 metres?" Clearly this requires that $x^2 + y^2 - 2x - 4y + 100 = 96$ or $x^2 + y^2 - 2x - 4y = -4$ and this defines a curve in a two-dimensional x-y plane. It can be rewritten (completing the squares) as: $(x-1)^2 + (y-2)^2 = 1$ so the points at elevation of 96 metres are located on this circle. Indeed, were we to ask "Where are all the points with elevation = H metres?" we'd get the circle $(x-1)^2 + (y-2)^2 = H - 95$, hence we could plot a variety of such curves for varying values of H.



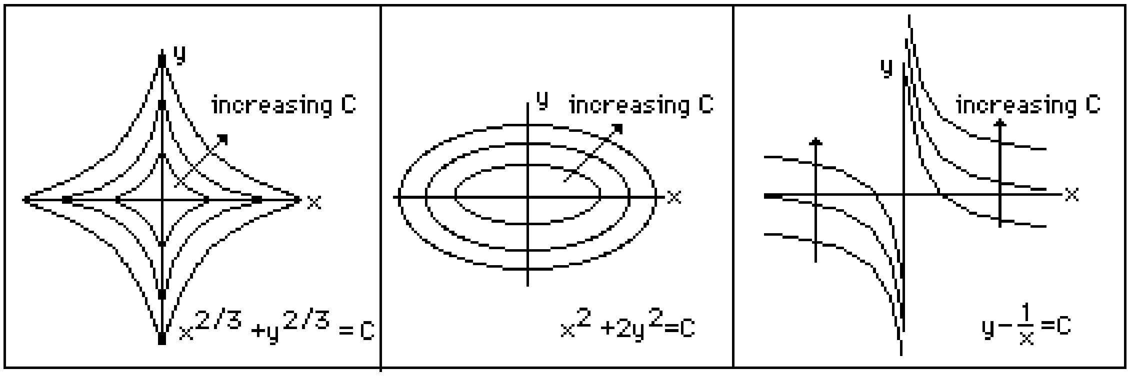
The curves shown are called LEVEL CURVES (for obvious reasons) and we've all seen such "topographical maps" ... although the level curves would rarely be circles! Note that this plot of level curves gives a good deal of information concerning the function of two variables $z = f(x,y) = x^2 + y^2 - 2x - 4y + 100$ and we didn't even have to move out of our comfortable 2-dimensional x-y plane. For example, the minimum elevation of 95 meters occurs at x=1, y=2 and the elevation increases in proportion to how far (x,y) is from (1,2) ... so the point (1,2) is sitting in a hole of sorts! Note that z might also be the temperature at the point (x,y), so these level curves would give the places where the temperature was 96˚ or 97˚ or 98˚ (z is presumably measured in degrees Fahrenheit). In this case the level curves might also be called "isothermal" curves. Maybe the curves describe the locations where the air pressure is equal; then they'd be called "isobars".

In general, for a given function of two variable, $z = f(x,y)$, the LEVEL CURVES are given by $f(x,y) = C$ for various values of the constant C.

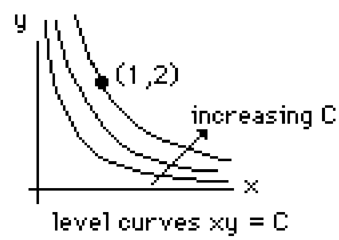
Example: Sketch the level curves for each of the following:

(a) $z = x^{2/3} + y^{2/3}$ (b) $z = x^2 + 2y^2$ (c) $z = y - \dfrac{1}{x}$

Solution:



In the sketches, we've indicated how the level curves change as C increases. For example, $z = y - \frac{1}{x} = C$ gives a set of hyperbolas which are shifted upward as C increases. The Astroid (as we've already seen) simply grows larger as C increases. So does the ellipse.



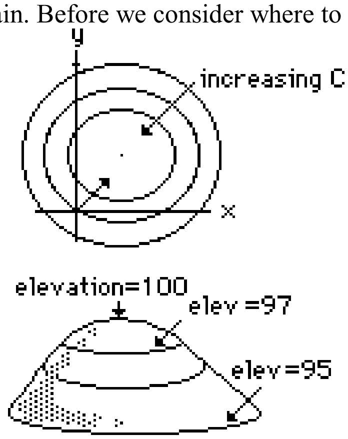
Example: For $z = xy$, determine the rate of change of y with respect to x on the level curve through the point $x = 1$, $y = 2$.

Solution: Level curves are $xy = C$ and differentiating implicitly gives $\frac{d}{dx}\, xy = 0$ or $x\frac{dy}{dx} + y = 0$ so $\frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{2}$ when $x = 1$, $y = 2$.

(Note: the level curve through this point is $xy = (1)(2) = 2$.)

Example: You are standing on the side of a mountain whose elevation is given by $z = 95 - x^2 - y^2 + 2x + 4y$ metres, where $x = 0$, $y = 0$ is your location, so $z = 95$ is your elevation. Sketch the level curves in your neighbourhood and determine in what direction you should climb so as to increase your elevation most rapidly.

Solution: The level curves $95 - x^2 - y^2 + 2x + 4y = C$ can be written $(x-1)^2 + (y-2)^2 = 100 - C$ so that the maximum elevation is $C = 100$ and it occurs at $x = 1$, $y = 2$, the "top" of the mountain, and (0,0) is on the side of the mountain. Before we consider where to climb, we sketch these curves (remembering that we are at the origin).



We want not only to climb in a direction of increasing C, but in the direction in which C increases most rapidly ... and a moment's thought (!) indicates that this means <u>perpendicular</u> to the level curve through (0,0), where we're located. Hence, we must find the slope of the level curve at (0,0) and move <u>perpendicular</u> to *that* direction. To find the level curve through (0,0) ... meaning the value of C ... put $x = 0$, $y = 0$ in $(x-1)^2+(y-2)^2 = 100 - C$ giving $5 = 100 - C$ so $C = 95$ (as we've already noted), hence the curve is:

$(x-1)^2+(y-2)^2 = 5$ and to obtain $\frac{dy}{dx}$ we simply differentiate to get

$2(x-1) + 2(y-2)\frac{dy}{dx} = 0$, then put $x = 0$, $y = 0$ and get $-2 - 4\frac{dy}{dx} = 0$,

then solve for $\frac{dy}{dx} = -\frac{1}{2}$. The negative reciprocal of this slope is the direction we want, so we head in a direction with slope = 2.

S: That's the stupidest thing I ever heard. I'd head directly toward the top of the mountain and that means directly toward (1,2) and that means in a direction with slope ... uh, the slope from (0,0) to (1,2) is $\frac{2-0}{1-0} = 2$ and as you can plainly see it's the same and I didn't need any derivatives and ...

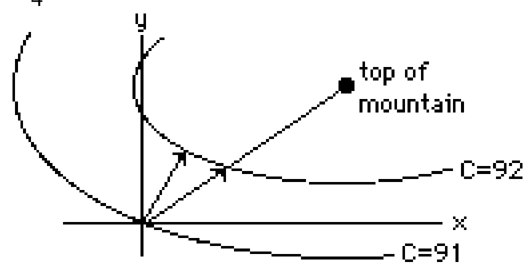
P: Okay, hold on. I picked a simple problem just so I could illustrate the technique. But what if the level curves were given by, say, $(x-1)^2 + 2(y-2)^2 = 100 - C$. Then what?

S: The place where the elevation is a maximum ... I mean, the top of my mountain ... is at x = 1, y = 2 and that's where I'd head ... same direction, same slope, namely 2.

P: Wrong! The curve through you ... I mean through (0,0), where you're standing, has $(0-1)^2 + 2(0-2)^2 = 100 - C$ so C = 91 so it's $(x-1)^2 + 2(y-2)^2 = 9$ and differentiating implicitly gives $2(x-1) + 4(y-2)\frac{dy}{dx} = 0$ and at (0,0) we'd get $-2 - 8\frac{dy}{dx} = 0$ so you should head in a direction with slope *perpendicular* to $\frac{dy}{dx} = -\frac{1}{4}$, hence your best bet is to climb with slope = 4, not 2.

S: Show me a picture.

P: Look at the picture ===>>>
You're at (0,0) with elevation C = 91 and <u>you</u> want to head directly for the top of the mountain at (1,2) which means to get to the higher elevation C = 92 you travel farther than if you had headed perpendicular to the level curve through (0,0). See?

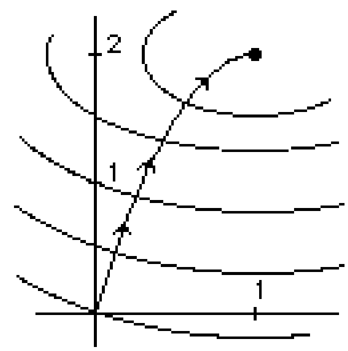


S: Yeah, but it means I'd have to ... uh, when I get to C = 92 I'd have to do it all over again, I mean find the tangent slope then the perpendicular slope and so on.

P: Right! But only if you wanted to climb the steepest slope at all times. Maybe you're a mountain climber and find it exhilarating and ...

S: Are you kiddin'? It'd take me an hour just to do the calculations. I like my way better. Besides, my way, I travel in a straight line. Your way you'd travel in some weird curve which is ... uh, sort of ...

P: Always perpendicular the level curves ... just like this ==>>
And you won't believe this but I can determine that path exactly just by taking the perpendicular direction to each level curve and inventing a differential equation $\frac{dy}{dx}$ = *something* which has this slope and I'd solve the DE and I could find the best path up the ...



S: And I'd be waiting at the top when you got there.

an Orthogonal Trajectory

Example:

If we move always in a direction perpendicular to the level curves $(x-1)^2 + 2(y-2)^2 = constant$, starting at x = 0, y = 0, what is the path? (Since it's orthogonal to the family of curves $(x-1)^2 + 2(y-2)^2 = constant$, it's called an ORTHOGONAL TRAJECTORY.)

Solution:

At any point (x,y) on a level curve the slope of the tangent line is obtained by implicit differentiation: $\frac{d}{dx}\left((x-1)^2 + 2(y-2)^2 = constant\right)$ gives $2(x-1) + 4(y-2)\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{1}{2}\frac{x-1}{y-2}$. If we move perpendicular to this direction we'd want to move so our slope is the negative reciprocal, namely $2\frac{y-2}{x-1}$. Hence our path would satisfy $\frac{dy}{dx} = 2\frac{y-2}{x-1}$ which (surprise!) is a separable differential equation. To solve, we separate the variables and integrate, giving: $\int\frac{dy}{y-2} = 2\int\frac{dx}{x-1}$ hence $\ln|y-2| = 2\ln|x-1| + C = \ln(x-1)^2 + C$ so $|y-2| = e^C e^{\ln|x-1|^2} = e^C (x-1)^2$ so $y - 2 = \pm e^C (x-1)^2$ or $y = 2 + K(x-1)^2$ where $K = \pm e^C$. The path is a parabola passing through (1,2) and, to pass through (0,0) as well, we'll need $0 = 2 + K(0-1)^2$ so K = -2. Hence out path will be:
$y = 2 - 2(x-1)^2$ or $y = 4x - 2x^2$ (and we check to see that it <u>does</u> pass through (0,0) and (1,2)).

S: That doesn't get you to the top any faster. In fact, my way ... which is heading straight for (1,2) ... I get there and I travel less distance. The shortest distance between (0,0) and (1,2) is a straight line! Didn't anybody ever tell you that?

P: Let's do something else.

3 Dimensional Surfaces

Although a graphical representation for z = f(x,y) can be obtained via LEVEL CURVES (which have the advantage of living in a familiar x-y plane), we can also pick x and y within the domain of the function "f", then calculate z, then plot a point (x,y,z) in a 3-dimensional rectangular space ... and we can do that for a host of values (x,y,z) and thereby generate a SURFACE. The coordinates (x,y,z) of every point on this surface would satisfy z = f(x,y).

Some typical ...

S: Wait! You said the "domain" of "f"?
P: Of course. It's just like functions of a single variable: y = f(x). We pick x from the domain of "f" and y must have a single unique value (else "f" isn't a *function*) and y lies in the "range" and we call x the independent variable and y the dependent variable. For functions of *two* variables, say z = f(x,y), x and y are the *two* independent variables and z is dependent upon them and we choose (x,y) from the domain of "f" and ..
S: Okay, okay ... keep going.

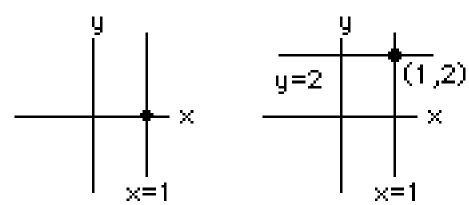
There are some 3-dimensional things we've seen before:

$$\boxed{x = x_0 + s\cos\alpha,\ y = y_0 + s\cos\beta,\ z = z_0 + s\cos\gamma}$$

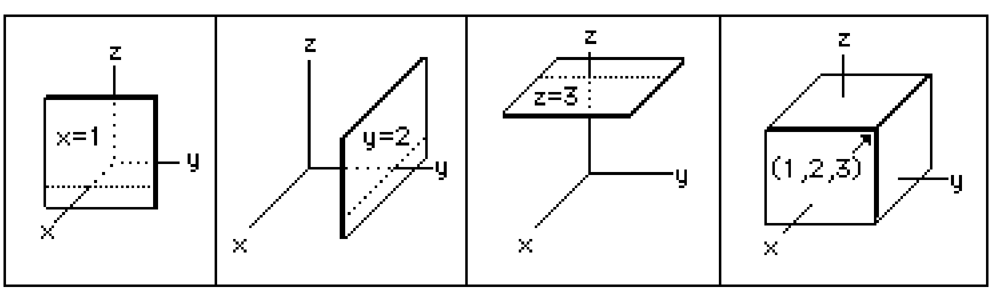
are parametric equations for a line through (x_0, y_0, z_0) in a direction which makes angles α, β and γ with the positive x- , y- and z-axes ... and "s" gives the distance from (x_0, y_0, z_0) to (x,y,z). In fact, distance in 2-D between (x_0, y_0) and (x_1, y_1) is given by the familiar expression $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$ and in 3-D the expression is a natural extension:

$$\boxed{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} = \text{distance between } (x_0, y_0, z_0) \text{ and } (x_1, y_1, z_1)}$$

It may seem that equations of 3-D surfaces are something new, but we already know many such equations. In fact, if we lived in a 1-D world, like the x-axis, then x = 1 would identify a single point P. However, in 2-D, all points (x,y) which satisfy x = 1 would lie on a straight line through P, parallel to the y-axis ... because x = 1 places no restrictions on the y-values so every y-value is possible so x = 1 describes an entire line when we move from 1-D to 2-D. In fact, we get the 2-D line x = 1 by starting with the 1-D point x = 1 and sliding the point parallel to the y-axis, sweeping out the line. In fact, in 2-D, *two* equations x = 1, y = 2 identify *two* lines and there is a single point which satisfies this pair of equations, and we denote this point by (1,2) ... which is no surprise.



Now consider the analogous situation in going from 2-D (which is familiar territory) to 3-D.



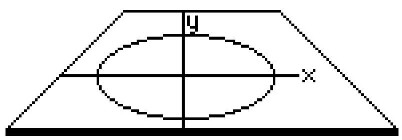
The 2-D equation x =1 representing a line places no restrictions on z so we slide this 2-D line (in the x-y plane) parallel to the z-axis and sweep out a 3-D plane: hence all points (x,y,z) which satisfy x = 1 lie on this plane.

Similarly, $y = 2$ and $z = 3$ are planes, and the *three* equations $x = 1$, $y = 2$, $z = 3$ identify *three* planes and there is a single point which satisfies this trio of equations and we denote this point by (1,2,3). Surprise!

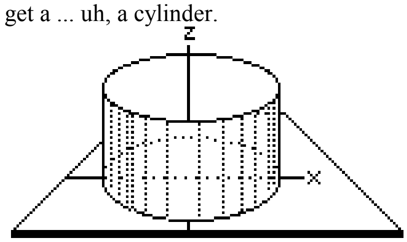
S: Does that mean I can just take curves I know, like $x^2 + y^2 = 1$ and $y = x^2$ and so on ... in the x-y plane, I mean ... and just slide them in the z-direction and get a surface?

P: Sure. Try it.

S: Okay, $x^2 + y^2 = 1$ is a circle and I slide it parallel to the z-axis and I get a ... uh, a cylinder.



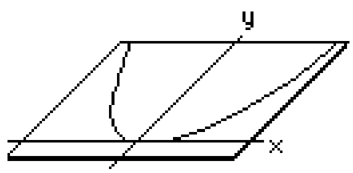
$x^2+y^2=a^2$ in 2D
a circle



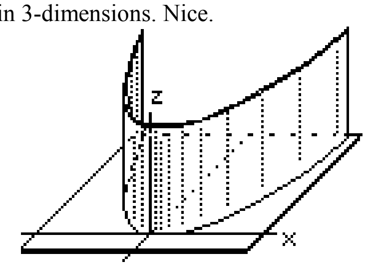
$x^2+y^2=a^2$ in 3-D
a circular cylinder

P: Right!. In fact, a right-circular cylinder. In fact, every time you take $y = f(x)$ and slide this 2-D curve parallel to the z-axis you <u>always</u> get what's called a "cylinder".

S: So $y = x^2$ is a parabola in 2-dimensions, but a parabola cylinder in 3-dimensions. Nice.



$y=x^2$ in 2-D
a parabola



$y=x^2$ in 3-D
a parabolic cylinder

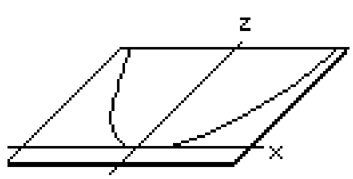
P: A parabolic cylinder.

S: So I already know thousands ... well, dozens of 3-D surfaces, like $y = x^3$ and $y = \sin x$ and $y = e^x$ and ...

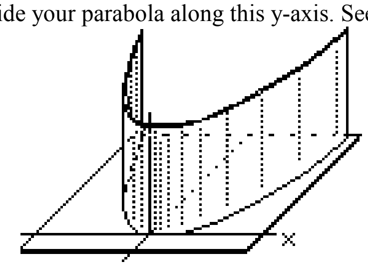
P: But they're all cylinders parallel to the z-axis. What about $z = x^2$? What's that?

S: Easy. I just ... uh, switch z and y and ... uh ...

P: First sketch $z = x^2$ in an x-z plane, then introduce a y-axis and slide your parabola along this y-axis. See?



$z =x^2$ in 2-D
a parabola



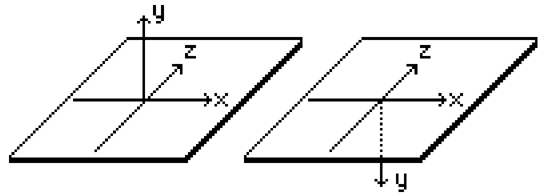
$z =x^2$ in 3-D
a parabolic cylinder

S: But which way does the y-axis go? I mean, up or down? Does it matter? I like up, myself.

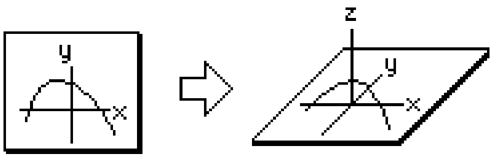
P: That's an interesting point. We have a choice and although it makes little difference we should pick one and agree on it and stick to it. In fact, a common convention is to pick the y-axis <u>down</u> as shown in the right half of the picture ===>>

S: That's really stupid, I mean ...

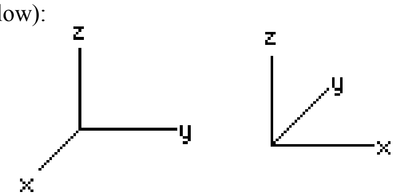
P: Hold on, let me show you the x-y-z coordinate system from several angles and see if you don't agree that it's a good convention.



Which way y?



First, it would be nice to consider a curve in a 2-D x-y plane then introduce a z-axis going up, as shown on the left. I think you'd agree with that. Then you can see that, with this convention, the 3 axes can be drawn in several ways (as shown below):

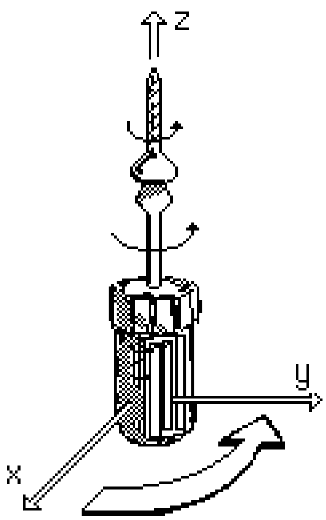


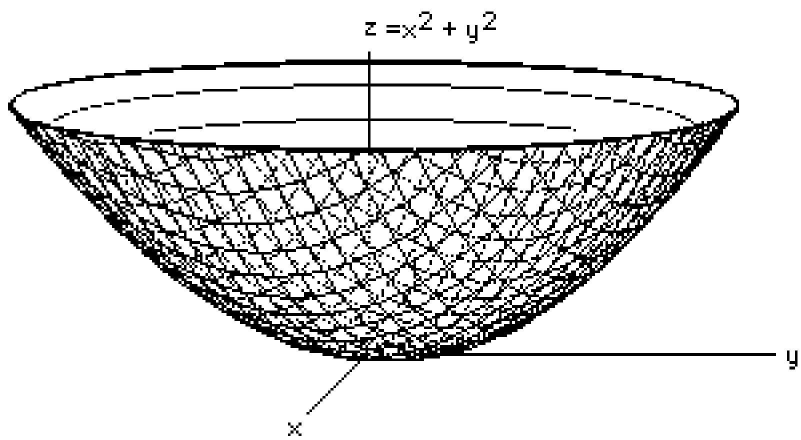
S: I can't really see what's what. What <u>is</u> the difference. Wouldn't my way be the same ... I mean ...

P: Okay, here's the convention. You rotate the x-axis into the y-axis and a *right-handed screw* should advance in the positive z-direction. <u>That's</u> how we pick the z-direction. <u>That's</u> the convention.

S: Picture?

P: Okay, here's a picture of the right-handed screw and also a picture of a surface, using that x-y-z coordinate system and you'll note that the surface <u>isn't</u> a cylinder. In fact, you can recognize a cylinder, one variable is missing.

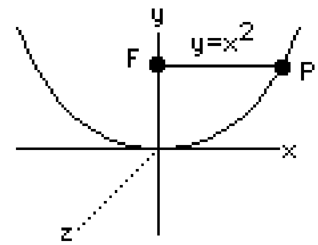




P: So what's $z = x^2 + y^2$?

S: A cereal bowl?

Revolving 2-D Curves to get 3-D Surfaces

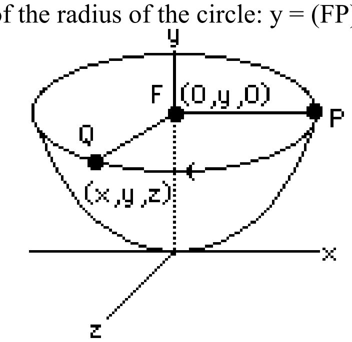


As well as sliding a 2-D curve $y = f(x)$ parallel to the z-axis and generating a 3-D "cylinder", we can also generate many interesting 3-D surfaces by revolving a 2-D curve about either axis. Let's start with the 2-D parabola $y = x^2$ and revolve it about the y-axis, sweeping out a surface. We want to find an equation satisfied by all points (x,y,z) which lie on the surface generated.

In 2-D, we consider the points F and P as shown ===>>

The relation $y = x^2$ really says that the y-value of P is the square of the distance FP (which, after all, is just x ... in this 2-D plane).

Now we revolve the parabola about the y-axis and note that F doesn't move but P sweeps out a whole *circle of points* and the y-values are all the same on this circle and equal to the square of the radius of the circle: $y = (FP)^2$.



If Q(x,y,z) is a typical point on this *circle of points*, then FQ = FP and since the y-value doesn't change we have $y = (FQ)^2$ but FQ is the distance from Q(x,y,z) to F(0,y,0) which is $\sqrt{(x - 0)^2 + (y - y)^2 + (z - 0)^2}$ = $\sqrt{x^2+z^2}$ so we get $y = \left(\sqrt{x^2+z^2}\right)^2$ or simply $\boxed{y = x^2 + z^2}$ which is the relation we want. In fact, every point (x,y,z) which satisfies $y = x^2 + z^2$ lies on this surface and because it was obtained by revolving a parabola, it's called ...

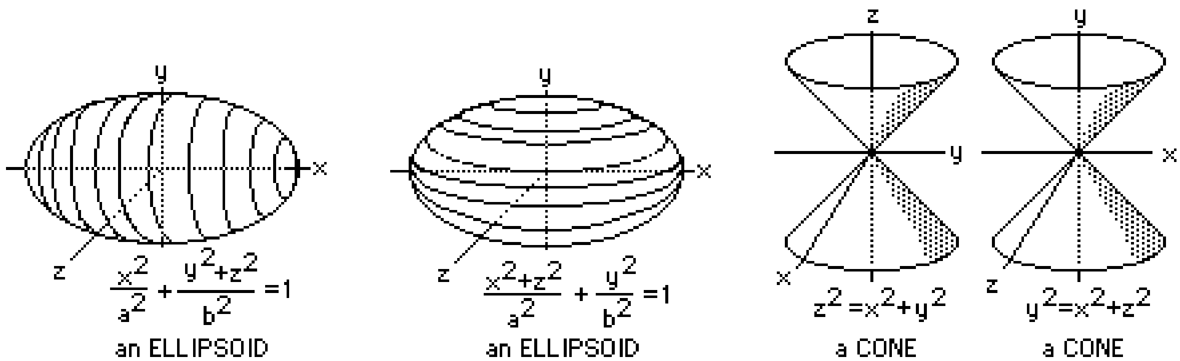
S: ... a parabolic cylinder!
P: It's <u>not</u> a cylinder! It's called a paraboloid and you've seen it before except it was written $z = x^2 + y^2$ because it was obtained by revolving a parabola in the x-z plane, namely, $z = x^2$, about ... you tell me, which axis?
S: Uh ... I'd say you started with $z = x^2$ and revolved about the z-axis.
P: Good. Let's find some simple rule so we can just write down the equation of the surface of revolution as soon as we're given the 2-D curve.

If we go carefully over the analysis given above for $y = x^2$ we see that there are two important things:

If we revolve about the y axis, then y doesn't change ... but x does, and the value of x^2 gets replaced by $x^2 + z^2$.

That means $y = x^2$ becomes $y = x^2 + z^2$

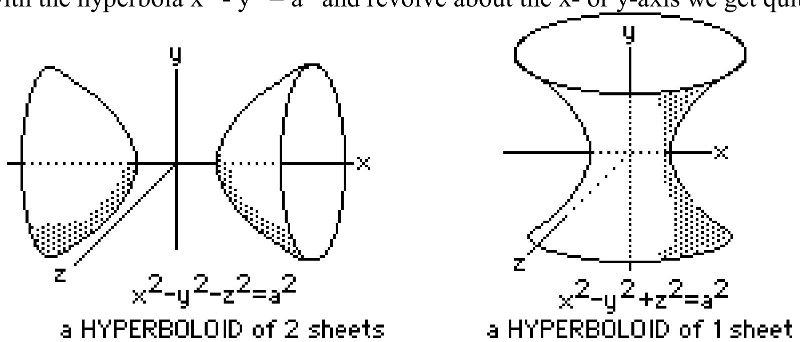
S: Hey! Let me do one!
P: Go right ahead.
S: Okay, I revolve $y = x^3$ about the y-axis and I get a 3-D surface whose equation is ... uh, I don't have any x^2, so what do I do?
P: Write $y = x^3$ as $y^2 = x^6 = (x^2)^3$.
S: Okay, then the surface is $y^2 = (x^2+ z^2)^3$. And if I revolve $y = x$ I first write it as $y^2 = x^2$ and I get $y^2 = x^2 + z^2$ and if I revolve ...
P: Hold on. What kind of surface do you think $y^2 = x^2 + z^2$ is?
S: Well, I revolved a line $y = x$ about the y-axis so I'd get ... uh, a cone. Right?
P: Good! Let me show you a few more ... but I'll rotate about different axes, but in each case if I rotate a curve in a plane about one axis, <u>that</u> variable doesn't change but the <u>other</u> does. If I revolve about the **y** axis, **y** doesn't change ... about the **z** axis and **z** doesn't change. Got it?



The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolved about the x-axis	The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolved about the y-axis	The line z = y about the z-axis	The line y = x about the y-axis

Note that an ellipse revolved about its major (longer) axis sweeps out a surface much like a hot dog bun. When revolved about its minor (shorter) axis, the surface is more like a hamburger bun.

If we start with the hyperbola $x^2 - y^2 = a^2$ and revolve about the x- or y-axis we get quite different surfaces:

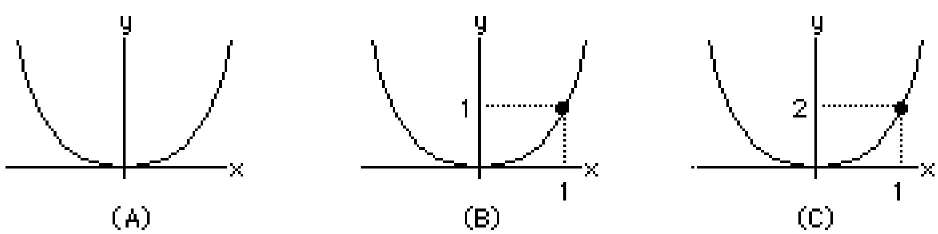


The hyperbola $x^2 - y^2 = a^2$ revolved about the x-axis	The hyperbola $x^2 - y^2 = a^2$ revolved about the y-axis

S: I hope you don't expect me to remember all this. Besides, you ask me to revolve a parabola or a straight line and so I *know* what the final surface looks like, but what if you just gave me $x^2 - 2y^2 + 3z^2 = 4$. What is it?

P: I replace the 2, 3 and 4 by 1 and get $x^2 - y^2 + z^2 = 1$, so it's a hyperboloid of 2 sheets.

S: Hey! You can't do that!



P: If I ask you to sketch $y = 2x^2$ in the x-y plane, you can just sketch $y = x^2$ then, since y is twice as large (it's $2x^2$ not x^2), you just stretch the curve in the y-direction ... but if you're only interested in the general shape you don't even bother to stretch. See? Look at the first graph above, labelled (A). Can you tell if it's $y = x^2$ or $y = 2x^2$? No! Not until I label some points. In (B) it's clear that it's $y = x^2$ whereas in (C) it's $y = 2x^2$. For $2x^2 + 3y^2 = 4$ you can sketch $x^2 + y^2 = 1$ (replacing 2 , 3 and 4 by 1) and get a circle then stretch it in each direction to get the ellipse $2x^2 + 3y^2 = 4$. In 3-D you do the same thing: given $x^2 - 2y^2 + 3z^2 = 4$ you imagine $x^2 - y^2 + z^2 = 1$ which is a hyperboloid, then you stretch in each direction and you <u>still</u> get a hyperboloid but now the cross-sections wouldn't be circles but rather ellipses. See?

S: But how much should I stretch it? I mean ...

P: It really isn't important if you just want to know what the surface looks like. You ask for $x^2 - 2y^2 + 3z^2 = 4$ and I sketch $x^2 - y^2 + z^2 = 1$ but if I don't put any tick marks on the axes (indicating the scale) you wouldn't know the difference. See?

S: But if you sketch $x^2+y^2=1$ I'd sure know it wasn't an ellipse!

P: Then I'm careful to make circles look like ellipses. Look again at the hyperboloids above. Can you really tell if the cross-sections are circles or ellipses? No. They could be graphs of $ax^2 - by^2 - cz^2 = d$ (for the 1-sheet type) or $ax^2 - by^2 + cz^2 = d$ (for the 2-sheet type) and "a", "b", "c" and "d" can be *any* positive numbers. Actually, what's important is the sign of the coefficients, not their size. See?

S: I guess ... but it ain't easy.

Some surfaces, of course, are <u>not</u> surfaces of revolution. For example $z = x^2 - y^2$ can NOT be obtained by revolving some 2-D curve about some line (like a coordinate axis). So what does it look like?

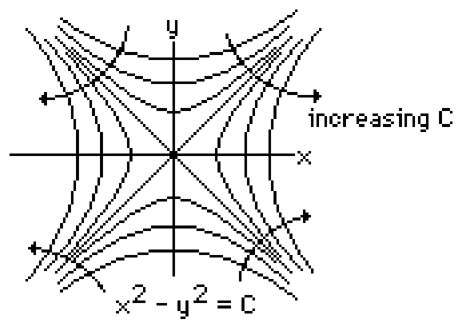
One way to see what the surface is like is to slice it by planes z = 1, z = 2, etc.. That gives us a bunch of cross-sections which are 2-dimensional and if we're lucky we can recognize them. For $z = x^2 - y^2$ all cross-sections with planes z = C gives $C = x^2 - y^2$ which are hyperbolas (in an x-y plane).

If this isn't sufficient to sketch the graph of $z = x^2 - y^2$ we can also slice by planes x = C so the cross-sections have the form $z = C^2 - y^2$: parabolas opening downward (in a y-z plane). Slicing the surface by planes y = C gives cross-section $z = x^2 - C^2$: parabolas opening upward (in an x-z plane). Because these sections are hyperbolas and parabolas the surface is called a **HYPERBOLIC PARABOLOID**.

This may seem confusing but ...

S: May <u>seem</u> confusing! It <u>is</u> confusing. Do you know how many cross-sections you've got? And how am I supposed to sketch them all in 3-D. I have a hard time in 2-D, with $x^2 - y^2 = C$!

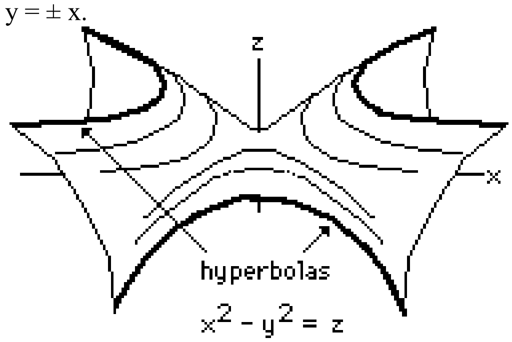
P: Pay attention.



To sketch $z = x^2 - y^2$ we slice the surface by planes z = C and get, for each such plane, a cross-section $x^2 - y^2 = C$ which we recognize as a hyperbola. We sketch them in the x-y plane and get ==>> and (surprise!) they're LEVEL CURVES! In fact, if $z = x^2 - y^2$ gives the elevation above sea-level at (x,y) then all points which have elevation z = C lie on the level curve $x^2 - y^2 = C$. We also note the direction of increasing C. Note that the level curve $x^2 - y^2 = C$ with $C < 0$ opens in the y direction whereas with $C > 0$ it opens in the x-direction. Finally, $x^2 - y^2 = 0$ is the pair of lines $y = \pm x$.

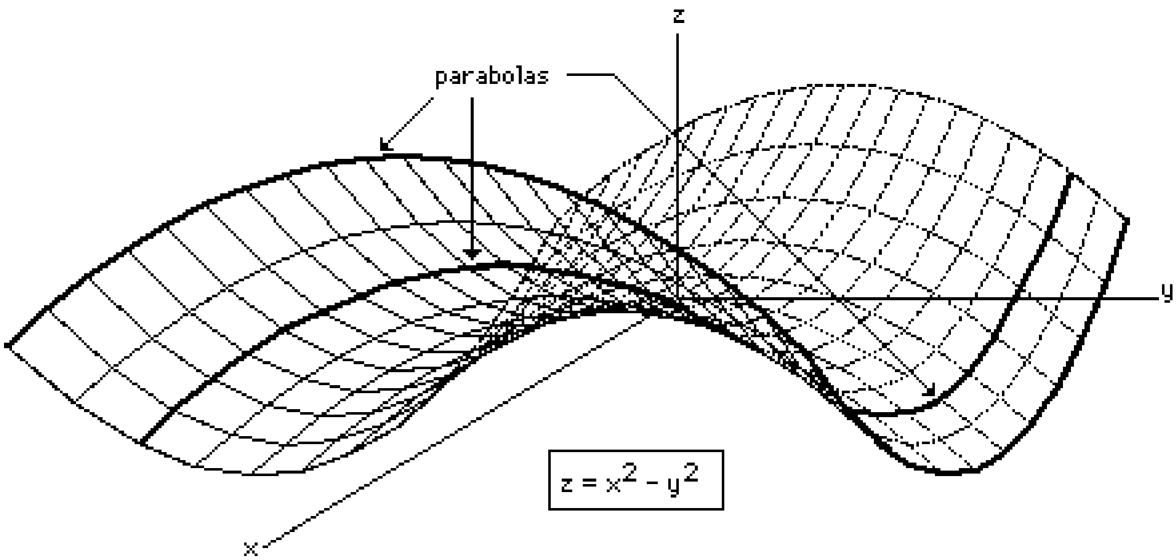
Now we tilt the x-y plane back a bit and introduce a z-axis and move these LEVEL CURVES by the amount C (in the +z direction if C > 0 and in the -z direction if C < 0).

This gives a rough sketch of the surface as shown ===>>>



Notice that when C < 0 the level curves are shifted down, below the x-y plane.

Below we show a more accurate, computer-plotted graph of $z = x^2 - y^2$.



In the above graph we've shown the various cross-sections with x = C and y = C, each of which is a parabola. The surface looks much like a saddle and indeed is often called a saddle surface. The origin lies at a point on the surface with peculiar characteristics. If the x-axis is east-west and the y-axis is north-south, then moving either east or west will increase your elevation and moving either north or south will decrease your elevation. Later when we consider how to find maxima and minima of functions of two variables, $z = f(x,y)$, we'll return to this surface. The origin appears to give a minimum if you move in the x-direction, but a maximum if you move in the y-direction. In fact, $x = y = 0$ gives neither a max nor a min but what is called (surprise!) a SADDLE POINT.

<u>LECTURE 14</u>

DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

To generalize the notion of the derivative of a function of a single variable, $y = f(x)$, at the place $x = a$, we'll repeat the procedure, imagining that we are standing at the point on the curve where $x = a$, $y = f(a)$. Now we change x by an amount Δx so the change in y is $\Delta y = f(a+\Delta x) - f(a)$ and we have $f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$.

We repeat this process in 3-space.

Suppose we are given a function of two variables $z = f(x,y)$ which we imagine as a surface in 3-dimensional space and we are standing at the point on the surface where $x = a$, $y = b$ so $z = f(a,b)$. Now we change *only* x by an amount Δx. The corresponding change in z is $\Delta z = f(a+\Delta x,b) - f(a,b)$ and we consider $\lim_{\Delta x \to 0} \frac{f(a+\Delta x,b) - f(a,b)}{\Delta x}$.

This is clearly a derivative of sorts although only x has changed. It is called the PARTIAL DERIVATIVE of $f(x,y)$ with respect to x, at the place $(x,y) = (a,b)$ and is denoted NOT by $\frac{df}{dx}$ as one might expect, but by $\frac{\partial f}{\partial x}$ ... the "∂" indicating immediately that there are other variables in "f". To indicate that the derivative is at $x = a$, $y = b$ we may write $\frac{\partial f}{\partial x}(a,b)$. In a similar manner we can define the PARTIAL DERIVATIVE with respect to y, at (a,b):

$$\frac{\partial f}{\partial x}(a,b) = \lim_{\Delta x \varnothing 0} \frac{f(a+\Delta x,b) - f(a,b)}{\Delta x} \qquad \frac{\partial f}{\partial y}(a,b) = \lim_{\Delta y \varnothing 0} \frac{f(a,b+\Delta y) - f(a,b)}{\Delta y}$$

There are other notations for these partial derivatives:

$\frac{\partial f}{\partial x}$ is also denoted by f_x and $\frac{\partial f}{\partial y}$ by f_y or, sometimes $\frac{\partial f}{\partial x}$ is denoted by f_1 and $\frac{\partial f}{\partial y}$ by f_2 where the subscripts mean "the first variable" and "the second variable", etc.. This is convenient if f is a function of 100 variables and you run out of letters in the alphabet so you just call the variables $x_1, x_2, \dots, x_{100}$ and then the notation $\frac{\partial f}{\partial x_{100}}$ is awkward so you just call it f_{100}.

It's clear that finding PARTIAL derivatives is no more difficult than finding ordinary derivatives: we just hold one variable fixed while we find the "ordinary" derivative with respect to the other variable.

Example: Determine each of the indicated partial derivatives:

(a) $\frac{\partial}{\partial x}\left(x^2 e^{-3y}\right)$ (b) $\frac{\partial}{\partial y}\left(\sin (xy^2)\right)$ (c) $\frac{\partial}{\partial z}\left(x^2 y + x^3 z^5\right)$

Solution:

(a) $\frac{\partial}{\partial x}\left(x^2 e^{-3y}\right) = 2xe^{-3y}$ (b) $\frac{\partial}{\partial y}\left(\sin (xy^2)\right) = \cos(xy^2)\,(2xy)$ (c) $\frac{\partial}{\partial z}\left(x^2 y + x^3 z^5\right) = 5x^3 z^4$

S: That's sneaky. The last one isn't a function of two variables, it's ...

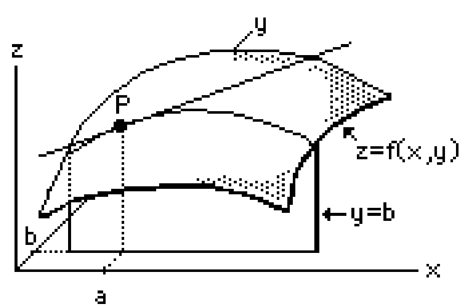
P: But see how simple it is? Find $\frac{\partial}{\partial z}$ of anything and you ignore all the other variables and concentrate on z alone and use the old stuff we've already learned about differentiating functions of one variable and ...

S: Yeah, I get it. Easy. But isn't there a picture?

Graphically speaking, if we hold y at the value "b" then the surface z = f(x,y) intersects the plane y = b in a curve which satisfies z = f(x,b), as shown ===>>>

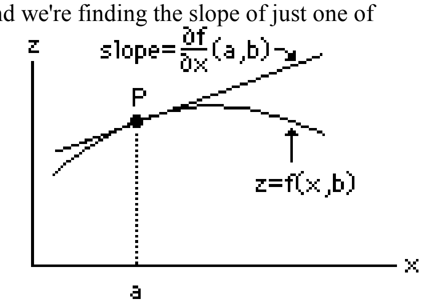
We are at point P(a,b,f(a,b)).

Now we change x from a to a + Δx and z changes from f(a,b) to f(a+Δx,b) and this gives two points on the curve of intersection and we then let Δx–>0 and get the limiting value of the rate of change $\frac{\Delta z}{\Delta x}$ and that's what we're calling $\frac{\partial f}{\partial x}(a,b)$.



S: And it's the slope of the tangent line, right?

P: Sort of. But there are jillions of tangent lines to the surface at the point P and we're finding the slope of just one of them and we have to be careful about the use of the word "slope". Since y is fixed and only x changes we can look at just the variation of z and x by standing way back along the y-axis and looking at this curve of intersection, and the z-axis is going up and the x-axis is going right and we get a picture like this ===>>>
Then we're back to a function of a single variable again and we know how to compute the derivative ... we just have to find the equation of the curve we see from this vantage point, and a moment's thought tells us that it's z = f(x,b).



S: So $\frac{\partial f}{\partial x}(a,b)$ <u>is</u> a slope, just like I said!

P: Yes, it's a slope, but pay attention: If I drew a line in the x-y plane and asked for its slope, you'd have no problem with that. You could take two points (x_1,y_1) and (x_2,y_2) on the line and compute $\frac{y_2 - y_1}{x_2 - x_1}$ to get the slope ... or you could measure the angle the line makes with the positive x-direction and compute the tangent of that angle. No problem. Now I draw a line in 3-D space and ask for the "slope". What do you do?

S: Easy! I take two points on the line and find the slope, just like you did in 2-D. Good, eh?

P: Here's two points... (x_1,y_1,z_1) and (x_2,y_2,z_2) ... compute the "slope" of the line joining them.

S: Uh ... well, let's see ... I give up.

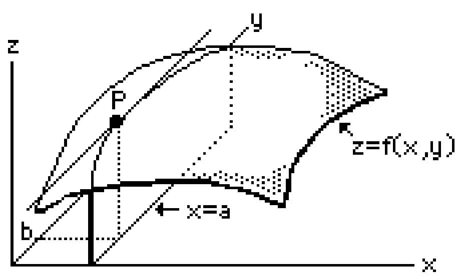
P: The point is, it's easy if you let only x change or only y change so you can stand back and view the line in 2-D with the z-axis going up and either the x- or y-axis going to the right, then you're familiar with this situation so you know what is meant by the "slope". But what happens if all three variables change? In fact, that's a problem not just for a line but for a surface. If we're at point P on the surface and only x changes then the rate of change of z = f(x,y) with respect to x is $\frac{\partial f}{\partial x}(a,b)$ ... but what if both x <u>and</u> y change. How rapidly does z change?

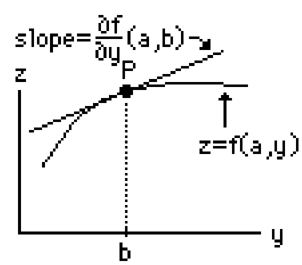
S: Will you be covering that?

P: Yes.

S: I was afraid you'd say that.

Now we'll change only y: at P (where x = a, y = b and z = f(a,b)), if x = a is fixed and *only y changes* then we move along a curve in our surface: the intersection of the surface z = f(x,y) with the plane x = a. Again we can view the relation between z and y from a vantage point a long way along the x-axis, looking back to see the z-axis going up and the y-axis going right and the curve of intersection and the point P and everything is back to 2-D and we have no problem with "what tangent line?" or "what slope?" The situation is shown below.





Example: The pressure of a gas P depends upon its volume V and temperature T according to PV = kT where k is a constant. If P = 1, V = 2 and T = 3, how rapidly is the pressure changing when V alone changes? Note: if P is measured in *pascals* and V in metre^3, then $\frac{\partial P}{\partial V}$ is measured in pascals/metre^3 (pa/m^3).

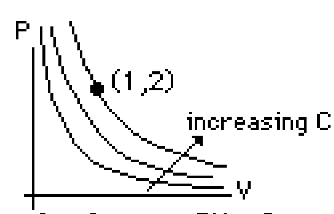
Solution: From $P = \frac{kT}{V}$ we get $\frac{\partial P}{\partial V} = -\frac{kT}{V^2} = -\frac{3k}{4}$ Pa/m^3 when V=2 and T=3. How to find k? We know that (P,V,T) = (1,2,3) satisfies PV = kT and that gives $k = \frac{(1)(2)}{3} = \frac{2}{3}$ so $\frac{\partial P}{\partial V} = -\frac{1}{2}$ pa/m^3.

S: That's not the way I'd do it. Since T isn't changing and neither is k then I'd just write PV = a constant and differentiate right away since I've got only two variables and I've done this kind of thing before so I'd use the product rule and I'd get $\frac{\partial}{\partial V}(PV)$ = 0 so $P\frac{\partial V}{\partial V} + V\frac{\partial P}{\partial V} = 0$ or $P + V\frac{\partial P}{\partial V} = 0$ and I'd plug in P = 1 and V = 2 and I'd get $\frac{\partial P}{\partial V} = -\frac{P}{V} = -\frac{1}{2}$. Good, eh?

P: Not bad ... in fact, very clever. And do you know that you never even used the fact that the temperature was T = 3 and you didn't have to find k either. Does this problem sound familiar? Does it say something to you?

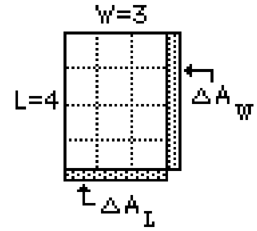
S: It says my solution if better than yours.

P: If PV = kT then, for a constant temperature, the relation between the pressure and volume is PV = C where the constant C = kT depends upon the temperature. See? You're on a LEVEL CURVE (as long as the temperature doesn't change ... so we can call it an *isothermal*). In fact, for varying temperatures, you get a whole family of such curves ... but you <u>know</u> that, because we've done this problem before when we talked about LEVEL CURVES, except we called the variables x,y.



level curves PV = C

Example: The area of a rectangular sheet is *length* = 4, *width* = 3. Does the area change more rapidly with *length* or with *width*?



Solution: Write A = W L where W is the *width* and L the *length*. Then, changing only the *width* W gives a rate of change $\frac{\partial A}{\partial W}$ = L while changing only the *length* L gives $\frac{\partial A}{\partial L}$ = W. Since W < L, then $\frac{\partial A}{\partial L} < \frac{\partial A}{\partial W}$ and the area changes more rapidly with W.

This is pretty obvious when you think of the changes in area of the rectangular sheet. A small increase in W increases the area (we can call it ΔA_W) more than does a small increase in L (which we call ΔA_L).

Example: A square box has sides of length x = 3 *cm*, and height y = 7 *cm*. If one of x or y is increased by a small amount, which will give the larger rate of increase in volume?

Solution: $V = x^2y$ and $\frac{\partial V}{\partial x} = 2xy = 2(3)(7) = 42$ *cm³/cm* whereas $\frac{\partial V}{\partial y} = x^2 = 9$ cm^3/cm so increasing x provides the greater rate of change of V.

Example: The cost per hat of making x hats in a month is $C = 6 - \frac{x}{500}$ *dollars/hat* (where the cost/hat is less if more are manufactured). The selling price per hat is $S = 9 + \cos\frac{\pi t}{6}$ *dollars/hat* which varies with the month t, t=0 being December and S = \$10.00/*hat*, an inflated Christmas price when the demand is high, and t = 6 being June where S = \$8.00/*hat* because sales of hats are poor in the summer ... and we assume that, at these prices, all x hats are sold each month. Investigate the total profit as a function of x, the number of items made in a month, and t, the month when they are sold. (Note that S is cyclic, or periodic, with a period of 12 months.)

Solution: The profit per hat is $S - C = 9 + \cos\frac{\pi t}{6} - \left(7 - \frac{x}{500}\right) = 2 + \cos\frac{\pi t}{6} + \frac{x}{500}$ *dollars/hat* which had better be positive, so even when S is a minimum (in June, when t = 6) we need $8 - (7 - \frac{x}{500}) > 0$ which it is, fortunately and that makes us happy. We really should check everything to see if this "mathematical model" is at all reasonable. So far it's okay ... but if we make more than 3000 hats in a month the cost per hat is zero ... which means our "model" isn't meant for large hat production! The total profit (for the t^{th} month) is:

$P(x,t)$ = (*number of hats*) (*profit/hat*) $= x\left(2 + \cos\frac{\pi t}{6} + \frac{x}{500}\right) = 2x + x\cos\frac{\pi t}{6} + \frac{x^2}{500}$ *dollars*.

Now consider what effect changing either x or t makes on our profit.

$\frac{\partial P}{\partial x} = 2 + \cos\frac{\pi t}{6} + \frac{x}{250}$ *dollars/hat* and $\frac{\partial P}{\partial t} = -\frac{\pi}{6}\, x \sin\frac{\pi t}{6}$ *dollars/month*. In December, t = 0 and we get $\frac{\partial P}{\partial x} = 3 + \frac{x}{250}$ which says the total profit increases at the rate of 3.00 *dollars/hat* plus an additional \$$\frac{1}{250}$ for each hat produced that month. Also $\frac{\partial P}{\partial t} = 0$ *dollars/month* so the profit isn't changing with time (in December, at least). In March, t = 3 and $\frac{\partial P}{\partial x} = 2 + \frac{x}{250}$ *dollars/hat* while $\frac{\partial P}{\partial t} = -\frac{\pi}{6}x$ *dollars/month* and is negative so the profit, P(x,t), is decreasing with time and this decrease is greater if more hats are made!

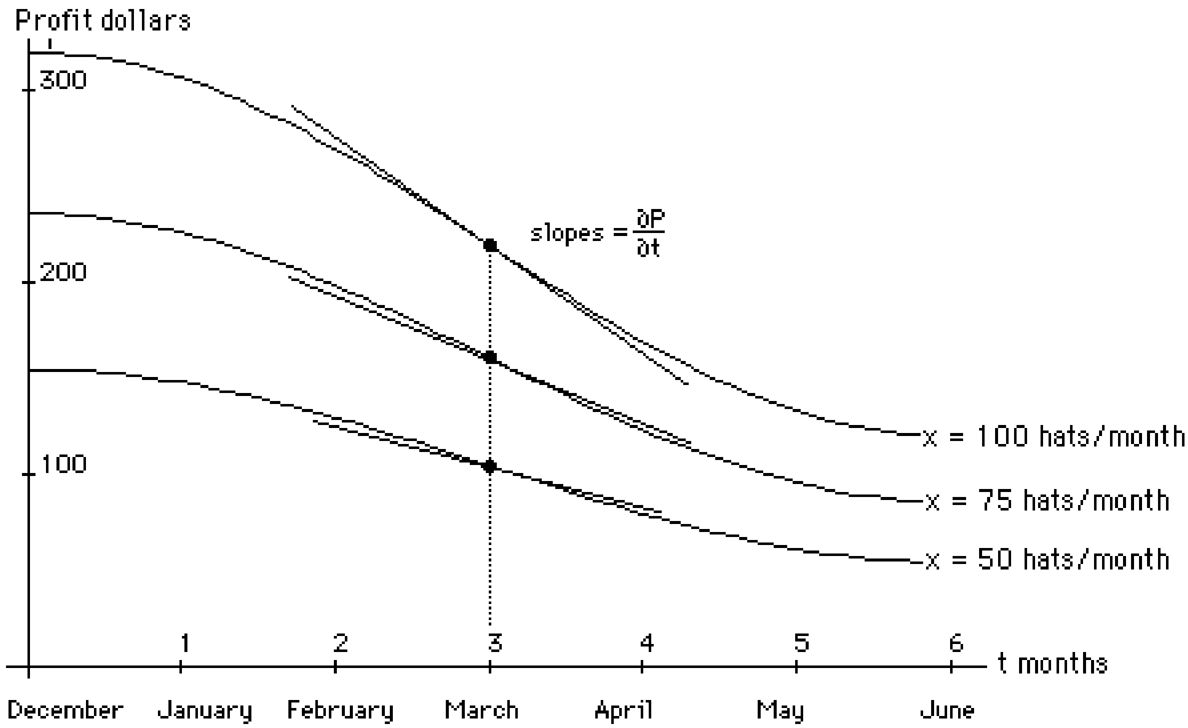
S: Aw, c'mon. You make more hats and the profits decrease? I mean ...

P: No, be careful. What's happening is that the profits per month, in the month of March, are negative. Not the profits but $\frac{\partial P}{\partial t}$, the *profit/month*. Now, it's this rate of change which depends upon how many hats are made that month: the more hats the greater the decrease per month. See? When you have functions of two variables the rate of change with respect to one may very well depend upon the other. All this makes sense, of course, because from December to June the total profits , $P(x,t) = 2x + x\cos\frac{\pi t}{6} + \frac{x^2}{500}$, go from $P(x,0) = 3x + \frac{x^2}{500}$ to $P(x,6) = x + \frac{x^2}{500}$ *dollars* so there is a big drop in profits namely, 2x *dollars* so you'd expect the rate at which P(x,t) decreases as t goes from t = 0 to t = 6 to be larger when x is larger ... because this drop is bigger ... and that's what the partial derivatives are telling us. See?

S: Not really.

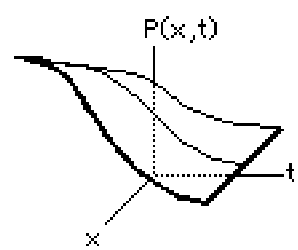
P: Then let me give you a picture. I'll assume the number of hats/month, x, is constant and see what happens to the profit as t goes from t = 0 to t = 6, then I'll change the number of hats produced per month and give you another graph of P versus t. That'll clear things up.

If we make x = 100 *hats/month* the profit is $P(100,t) = 220 + 100 \cos \frac{\pi t}{6}$ and if we make x = 50 hats/month the total profit is $P(50,t) = 120 + 50 \cos \frac{\pi t}{6}$ *dollars* (for month number "t") and so on. The graphs look like so:



You'll notice that there is a drastic decrease in profits from December to June when you make lots of hats each month, and that's illustrated by the value of $\frac{\partial P}{\partial t}$ which becomes more negative as x increases.

S: I knew a picture would help. Like I always say, a picture is worth a thousand ...
P: Right! Now let's go on.
S: Wait! How about a picture of P(x,t) in 3-D?
P: Okay, here's a rough sketch ===>>>
See how, for each x-value, the curves are cosine functions of t which decrease more rapidly as the x-value increases?

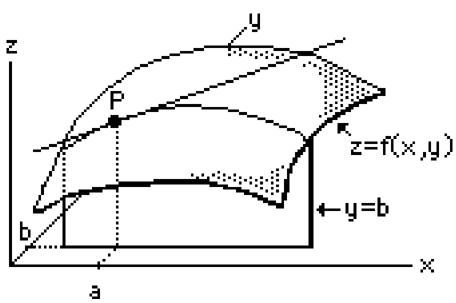


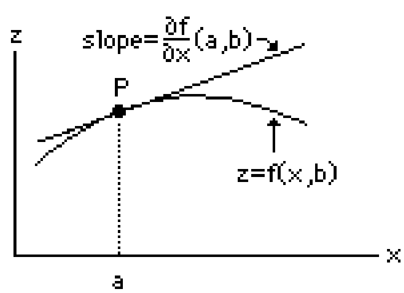
S: Now suppose we're sitting at a particular point on this surface. That means a particular t-month and a particular x-quantity of hats/month. Where do we go so P(x,t) increases most rapidly? I mean ...
P: I know exactly what you mean. You want to maximize your profits so you'd like to know how to modify x and t to accomplish this. Like being on the side of a mountain and asking "Which direction to increase my elevation most rapidly?" Good luck. The variable "t" isn't something you can change, else you'd arrange for it to be Christmas all the time.
S: But isn't "t" called an independent variable and doesn't that mean I can change it at will?
P: No. It means it's independent and does what it pleases. Time marches on ... which reminds me, we should too.
S: I notice that you keep trying to put everything into a 2-dimensional plane ... the x-y plane or maybe the x-z plane or something ... and you rely on what we've done before to find derivatives because you've got a function of just one variable because you're keeping one of your variables constant so ...
P: Yes, yes ... of course. We'd like to reduce the problem to one we've already solved. Did I ever tell you the story of the mathematician and the engineer?

S: Yes, yes ... of course. Anyway, I presume that we can take the derivative of the derivative and get the second derivative, right?
P: Right. Now pay attention:

HIGHER PARTIAL DERIVATIVES

If we hold y fixed and concentrate only on the change in x then $\frac{\partial f}{\partial x}$ is the rate of change of f with respect to x and we can evaluate it at any point P(a,b). If we leave $\frac{\partial f}{\partial x}$ as a function of x and y (without substituting x = a, y = b) then we can differentiate again: $\frac{\partial}{\partial x}\frac{\partial f}{\partial x}$ which is also denoted by $\frac{\partial^2 f}{\partial x^2}$. This is the second partial with respect to x, and we can also evaluate it at P(a,b). In a similar manner we can consider $\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$, the second partial with respect to y. Each has its geometric significance.





Consider the surface described by z = f(x,y) and the plane y = b. This plane intersects the surface in a curve which, when viewed from a long way along the negative y-axis (so we see the z-axis going up and the x-axis going right), is described by z = f(x,b). Whereas the slope of this curve is $\frac{\partial f}{\partial x}(x,b)$, hence is $\frac{\partial f}{\partial x}(a,b)$ when x = a, at the point P(a,b), we can see that $\frac{\partial^2 f(x,b)}{\partial x^2}$ gives the rate of change of this slope as x changes along the curve, and the sign of $\frac{\partial^2 f(x,b)}{\partial x^2}$ will indicate whether the curve is concave up or down. If we substitute x = a we'll get $\frac{\partial^2 f}{\partial x^2}(a,b)$, hence an indication of the "concavity" at P(a,b).

Of course, this is not the concavity of the <u>surface</u> z = f(x,y), but only of this particular <u>curve</u> on that surface. Indeed, it's an interesting question to ask: "What does one mean by the concavity at a point on a surface, z = f(x,y)?"

Of course, we can repeat this for another curve: the intersection of the plane x = a with z = f(x,y). Then when viewed from a mile or two up the positive x-axis, looking back at the z-axis going up and the y-axis going right, we see the curve z = f(a,y) and $\frac{\partial f}{\partial y}(a,y)$ and $\frac{\partial^2 y}{\partial x^2}(a,y)$ give the slope and its rate of change at any y-value along <u>this</u> curve.

Sometimes this geometrical interpretation of the second partials is useful ... provided you only change <u>one</u> of the variables.

S: But I'd like to know what happens if <u>both</u> x and y change, together. What's the "slope" and what's the rate of change of z = f(x,y)?
P: I'm glad you asked that question ...

<u>LECTURE 15</u>

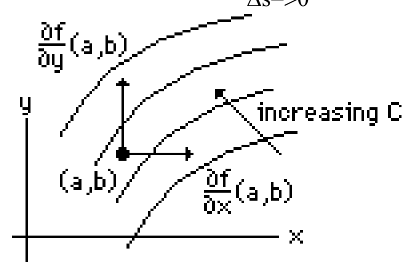
DIRECTIONAL DERIVATIVES

Suppose we have z = f(x,y) and we're at the point x = a, y = b so z = f(a,b). Now we change x by Δx and y by Δy, then z becomes f(a+Δx,b+Δy) hence changes by Δz = f(a+Δx,b+Δy) - f(a,b). What do we mean by the "rate of change of z"? It's not the rate of change with respect to x or y alone, but with respect to ... both, somehow.

In fact, just as we climbed a mountain whose elevation was given by $z = f(x,y)$, we'd want to know how quickly z changes with change in our x-y position. We'd take the ratio: $\frac{\text{change in z}}{\text{change in x-y position}}$ then let the change in x-y position go to zero. What do we mean by "change in x-y position"? We mean the distance between (a,b) and $(a+\Delta x, b+\Delta y)$ and that's $\sqrt{\Delta x^2 + \Delta y^2}$ so we'd consider the ratio: $\frac{\Delta z}{\sqrt{\Delta x^2 + \Delta y^2}}$ and we'd let $\sqrt{\Delta x^2 + \Delta y^2}$ –>0.

Although that's what we want, we should find an easier way to compute this limit! In particular, since $\sqrt{\Delta x^2 + \Delta y^2}$ is a terrible expression to deal with and since it's just the distance in the x-y plane we should give it a name, and because it's small (because Δx and Δy are small), the name should reflect this, so we'll call it Δs.

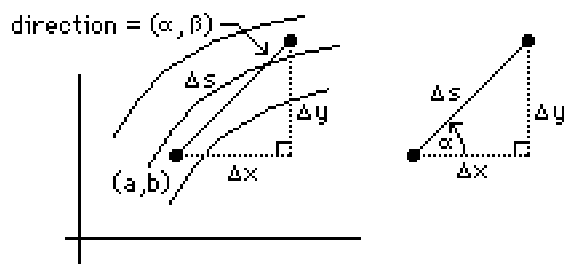
So far we've moved a distance $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$ and the change in z is Δz and we want $\lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s}$.



Although x <u>and</u> y are both changing, we can <u>still</u> work in the x-y plane by using LEVEL CURVES again.

In the diagram are some level curves for $z = f(x,y)$, namely $f(x,y) = C$ for various values of C. We're at (a,b) and if we move in the x-direction then the rate of change of z is the limit as Δx–>0 of $\frac{\text{change in z}}{\text{change in x position}} = \frac{\Delta z}{\Delta x}$ and that's $\frac{\partial f}{\partial x}(a,b)$. If we move in the y-direction (by an amount Δy), the rate of change: $\frac{\partial f}{\partial y}(a,b) = \lim_{\Delta y \varnothing 0} \frac{\Delta z}{\Delta y}$.

Now we move in an arbitrary direction given by the angles (α,β) where α is the angle our direction makes with the positive x-axis and β is the angle with the positive y-axis. (Remember that? ... when we discussed parametric equations of lines?)



In the diagram, we've moved from (a,b) a distance $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$ and we want to investigate the limiting value of $\frac{\Delta z}{\Delta s} = \frac{f(a+\Delta x,b+\Delta y) - f(a,b)}{\Delta s}$ as Δs–>0. To this end we consider the changes in x and y in turn, writing

$$\frac{\Delta z}{\Delta s} = \frac{f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)}{\Delta s} + \frac{f(a+\Delta x,b) - f(a,b)}{\Delta s}$$

where the first fraction has only y changing and the second only x changing.

S: Hold on! How did you know to add and subtract like that! I mean ...

P: I'm familiar with functions of a single variable, so I let the variables change one-at-a-time. I'm at x = a, y = b and I want to get to $x = a+\Delta x$, $y = b+\Delta y$. First I change x, going from (a,b) to $(a+\Delta x,b)$ and find the change in z: $f(a+\Delta x,b) - f(a,b)$. Now I change y, going from $(a+\Delta x,b)$ to $(a+\Delta x,b+\Delta y)$ and there's another change in z: $f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)$. The total change is $\big(f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)\big) + \big(f(a+\Delta x,b) - f(a,b)\big)$ which, of course, is just $f(a+\Delta x,b+\Delta y) - f(a,b)$.

Let's look at just the second fraction. If only we had $\frac{f(a+\Delta x,b) - f(a,b)}{\Delta x}$ we could take the limit and recognize it as $\frac{\partial f}{\partial x}(a,b)$ ... but we can write the second fraction as $\frac{f(a+\Delta x,b) - f(a,b)}{\Delta x}\frac{\Delta x}{\Delta s}$ and note that $\frac{\Delta x}{\Delta s}$ is just $\cos\alpha$, so now we can let Δs–>0 and get $\lim_{\Delta s \varnothing 0} \frac{f(a+\Delta x,b) - f(a,b)}{\Delta x}\frac{\Delta x}{\Delta s} = \frac{\partial f}{\partial x}(a,b)\ \cos\alpha$.

We repeat this procedure for the first fraction, writing:

$$\frac{f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)}{\Delta s} = \frac{f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)}{\Delta y}\frac{\Delta y}{\Delta s} = \frac{f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)}{\Delta y}\ \sin\alpha$$ and now let Δy–>0

so that $\dfrac{f(a+\Delta x,b+\Delta y) - f(a+\Delta x,b)}{\Delta y} \to \dfrac{\partial f}{\partial y}(a+\Delta x,b)$ ... that's the definition of the partial derivative at the point x = a+Δx, y = b. Now let Δx–>0 as well, so $\dfrac{\partial f}{\partial y}(a+\Delta x,b) \to \dfrac{\partial f}{\partial y}(a,b)$ and having let both Δx–>0 and Δy–>0 we also have Δs–>0 so we're finished.

S: Huh? Finished? Finished with what?
P: Don't you see? We've now got the rate of change of z in <u>any</u> direction. You don't have to move in the x-direction or the y-direction. You can pick your direction, like north-by-north-east, and that determines the angle α and hence you can compute the rate of change in this direction.
S: And what is it?
P: We need a nice notation for it which looks similar to $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ except it has to have distance in the α-direction, not just the x- or y-direction. Got any good ideas? Remember that it has to have the dimensions $\dfrac{\text{dimensions of f}}{\text{dimensions of s}}$.
S: Yeah, call if $\dfrac{\partial f}{\partial s}$.

the DIRECTIONAL DERIVATIVE

$$\boxed{\frac{\partial f}{\partial s}(a,b) = \frac{\partial f}{\partial x}(a,b)\cos\alpha + \frac{\partial f}{\partial y}(a,b)\sin\alpha}$$

where "s" is distance measured in a direction which makes an angle α with the positive x-axis.

Some observations about the directional derivative:

- If α = 0, we have the positive x-direction and the above formula gives $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}$ as we'd expect.
- If $\alpha = \dfrac{\pi}{2}$, we have the positive y-direction and we find that $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial y}$.
- For other directions, the rate of change is a combination of each of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, the latter being weighted more heavily if α is closer to $\dfrac{\pi}{2}$, and so on.
- If we stand fixed at a point x = a, y = b, then $\dfrac{\partial f}{\partial x}(a,b)$ and $\dfrac{\partial f}{\partial y}(a,b)$ are fixed numbers and only cos α and sin α change, and $\dfrac{\partial f}{\partial s}$ takes the form A cos α + B sin α (where A and B are constants) and presumably we could determine the value of α which makes $\dfrac{\partial f}{\partial s}$ a maximum or minimum ... and we'll do just that in the next lecture. (It's like standing on the side of a mountain and determining the direction in which the elevation increases most rapidly).
- Since cos α and sin α are dimensionless (think of them as the ratio of sides of a triangle), the dimensions of $\dfrac{\partial f}{\partial s}$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are the same : if f is in *hectares* and s, x and y are in *years*, then each has the dimensions of *hectares/year*.

Example: Determine the rate of change of the given function at the indicated point, in the given direction.

(a) $f(x,y) = x^2 + x \sin y$ at (1,π) in the direction which makes an angle of $\dfrac{\pi}{3}$ with the positive x-axis.

(b) $f(x,y) = e^{2x} + xy$ at (0,1) in the direction which makes an angle of $-\dfrac{\pi}{4}$ with the positive x-axis.

(c) $f(x,y) = x^3y$ at (0,0) in the negative y-direction.

(d) $f(x,y) = x^3y$ at (0,0) in the negative x-direction.

Solution:

(a) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha$ where $\alpha = \frac{\pi}{3}$, and $\frac{\partial f}{\partial x} = 2x + \sin y = 2$ and $\frac{\partial f}{\partial y} = x\cos y = -1$ at $(1,\pi)$.

Hence $\frac{\partial f}{\partial s} = (2)\cos\frac{\pi}{3} + (-1)\sin\frac{\pi}{3} = 2\frac{1}{2} - \frac{\sqrt{3}}{2} = 1 - \frac{\sqrt{3}}{2}$.

(b) $\frac{\partial f}{\partial x} = 2e^{2x} + y = 3$ and $\frac{\partial f}{\partial y} = x = 0$ at (0,1), hence $\frac{\partial f}{\partial s} = (3)\cos(-\frac{\pi}{4}) + 0 = \frac{3\sqrt{3}}{2}$.

(c) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha$ where $\alpha = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$ (or any other equivalent angle) .

At (0,0) we have $\frac{\partial f}{\partial x} = 3x^2y = 0$ and $\frac{\partial f}{\partial y} = x^3 = 0$ so $\frac{\partial f}{\partial s} = 0$.

(d) $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha$ where $\alpha = \pi$ and at (0,0) we have $\frac{\partial f}{\partial x} = 3x^2y = 0$ and $\frac{\partial f}{\partial y} = x^3 = 0$ so $\frac{\partial f}{\partial s} = 0$.

It seems clear that $\frac{\partial f}{\partial s} = 0$ at (0,0) regardless of the direction we move, since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero at (0,0). The surface $z = x^3y$ must be very flat at the origin.

S: A picture, please.

P: Okay, but let's predict what $z = x^3y$ will look like. First off we note that it rises from the origin when (x,y) goes into the first quadrant where $x>0$ and $y>0$ ...

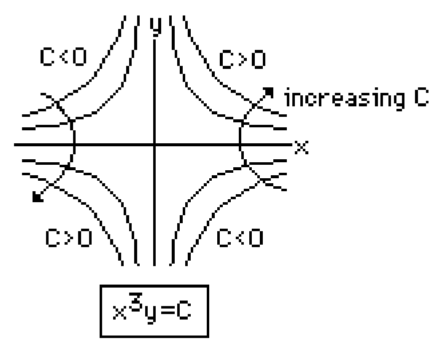
S: ... and into the third quadrant too ... and it falls into the other quadrants ...

P: What about the level curves. Do they help?

S: Uh, sure, why not. Do some.

P: <u>You</u> do some.

S: Well ... I'd write $z = x^3y$ and let $z = C$ so I'd get LEVEL curves $x^3y = C$ and I'd write $y = \frac{C}{x^3}$ and I'd do a Quick&Dirty sketch ... uh, for small x it looks like ... uh ...



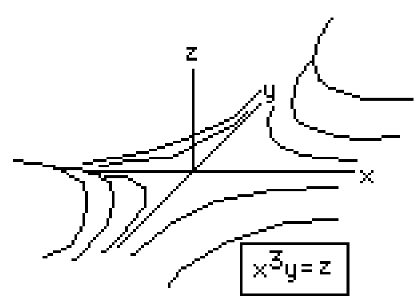
P: It's too simple for Q&D: $y = \frac{C}{x^3}$ has a vertical asymptote at $x = 0$ with

$\lim_{x\to 0^+} y = \infty$ (assuming C>0) and a horizontal asymptote at $y = 0$ since

$\lim_{x\to\infty} y = 0$ and it's an ODD function and that's enough to sketch a few curves. Of course, before sketching $z = x^3y$ we might also note that if we slice this surface with a plane $y = A$ we'd get a cubic cross-section: $z = Ax^3$ and if we slice it with a plane $x = B$ we'd get a linear cross-section: $z = B^3y$ and what would we get if we slice it with a plane $z = C$?

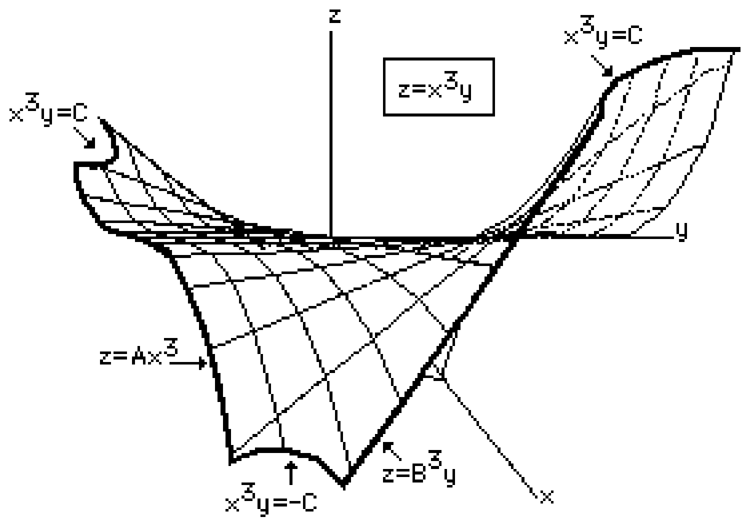
S: Huh? I haven't the foggiest idea.

P: We'd get the level curves, of course: $x^3y = C$ (where C may be positive or negative). If we just rely on the level curves we might sketch $z = x^3y$ like so ===>>>



S: What about the big picture ... the big computer picture?

P: Okay, here's a computer-plotted picture ... below:



Note, in the diagram, the intersection with the plane y = A (namely $z = Ax^3$), has A < 0 so the curve of intersection ... when viewed with the z-axis going up and the x-axis going right ... looks like $z = - x^3$ (not like $z = x^3$). Note, too, that the entire x-axis lies on this surface since points on the x-axis have coordinates (x,0,0) which clearly satisfies $z = x^3y$ for any x-value. Also, the entire y-axis, containing points (0,y,0), satisfies the equation $z = x^3y$ as well.

The "flatness" of the surface, near the origin, is evident, so it's not surprising that the rate of change of z is zero in ANY direction.

Example: The density of ants at a location (x,y) is given by $D(x,y) = K \frac{e^{-x}}{1+y^2}$ *ants per metres²* where x and y measure distance (in *metres*) from the queen ant who is located at (0,0): x is east-west and y is north-south distance. What is the rate of change of ant density at (0,0), in a north-west direction?

Solution: $\frac{\partial D}{\partial s} = \frac{\partial D}{\partial x} \cos \alpha + \frac{\partial D}{\partial y} \sin \alpha$ where $\alpha = \frac{3\pi}{4}$ and $\frac{\partial D}{\partial x} = - K \frac{e^{-x}}{1+y^2} = - K$ and $\frac{\partial D}{\partial y} = - K \frac{2y\, e^{-x}}{(1+y^2)^2} = 0$ at (0,0). Hence $\frac{\partial D}{\partial s} = (-K) \cos \frac{3\pi}{4} + 0 = \frac{K}{\sqrt{2}}$ *ants/metres³*.

S: *Ants per cubic metre*? You're kidding?

P: Not at all. If D is measure in *ants per metres²* and s is in *metres*, then $\frac{\partial D}{\partial s}$ is in $\frac{ants/m^2}{m}$ = *ants/m³*. Of course, we might also say *ants/m² per m*. See?

S: Sounds like you got so many ants per ... uh, volume ... or something. Anyway, you're talking about ants because you think I'll be impressed with how useful this is ... but I'm not. How on earth would you get $D(x,y) = K \frac{e^{-x}}{1+y^2}$? If I'm a biologist can I expect somebody to hand me this function and say "find the rate of ...

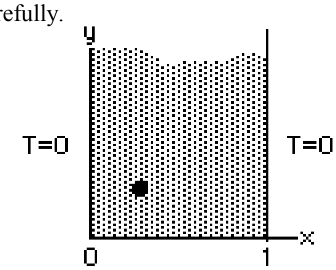
P: Pay attention.

Example: You are standing at the origin (0,0) and the temperature is given by $T(x,y) = 100\, e^{-x} \cos y$ °C at location (x,y), each of x and y being measured in *kilometres*. In what direction should you move so it gets cooler most quickly?

Solution: We interpret "cooler most quickly" to mean that the rate of change of temperature should be as negative as possible, in the optimal direction. If "s" measures distance in the direction α then we'd want $\frac{\partial T}{\partial s}$ to be as negative as possible. We have that $\frac{\partial T}{\partial s} = \frac{\partial T}{\partial x} \cos \alpha + \frac{\partial T}{\partial y} \sin \alpha = (- 100\, e^{-x} \cos y) \cos \alpha + (- 100\, e^{-y} \sin y) \sin \alpha$ which, at our location (0,0), is $\frac{\partial T}{\partial s} = - 100 \cos \alpha + 0 = - 100 \cos \alpha$. We want $\alpha = 0$ so $\frac{\partial T}{\partial s} = -100$ *°C/kilometre*.

S: Sure, sure. Move one km due east and the temperature drops from boiling to freezing ... and this is *useful*?

P: Pay attention. This is only the rate of change at (0,0), in the x-direction. It changes, you know. In fact, T never reaches 0 on the x-axis because, putting y = 0, $T(x,0) = 100\ e^{-x}$ °C. However, it does get to 0°C when $y = \frac{\pi}{2}$. Besides, $T(x,y) = 100\ e^{-x} \cos y$ is only an invention ... so we can practice taking derivatives and interpret rates of change in various directions. Perhaps we can invent a more reasonable temperature variation, and study it more carefully.



Example: The temperature of a plate lying in $0 \le x \le 1$, $y \ge 0$ is given by $T(x,y) = \sin \pi x\ e^{-\pi y}$. In what direction, from $(\frac{1}{4}, \frac{1}{4})$, is the temperature increasing most rapidly?

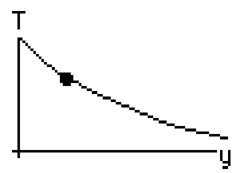
(The plate is shown in the diagram, together with the point.)

Solution: If "s" measures distance in the α-direction, then $\frac{\partial T}{\partial s} = \frac{\partial T}{\partial x} \cos \alpha + \frac{\partial T}{\partial y} \sin \alpha$ gives

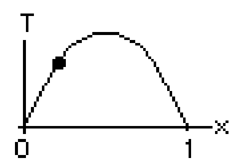
$\frac{\partial T}{\partial s} = (\pi \cos \pi x\ e^{-\pi y}) \cos \alpha + (-\pi \sin \pi x\ e^{-\pi y}) \sin \alpha = \frac{\pi}{\sqrt{2}} e^{-\pi/4} \cos \alpha - \frac{\pi}{4} e^{-\pi/4} \sin \alpha = \frac{\pi}{4} e^{-\pi/4}(\cos \alpha - \sin \alpha)$ at the point $(\frac{1}{4}, \frac{1}{4})$. Our problem is to choose α so that this rate of change is maximized. Since it's a function of a single variable, α, and since we can assume α lies in the <u>closed</u> interval $[-\pi,\pi]$ ... which clearly includes every possible direction ... we can just find the critical points of $(\cos \alpha - \sin \alpha)$ in this interval and evaluate this function there as well as at the endpoints $\alpha = -\pi$ and $\alpha = \pi$ and pick the largest of these values. We have:

$\frac{d}{d\alpha}(\cos \alpha - \sin \alpha) = -\sin \alpha - \cos \alpha = 0$ when $\tan \alpha = -1$ so $\alpha = -\frac{\pi}{4}$ or $\alpha = \frac{3\pi}{4}$ so we evaluate $(\cos \alpha - \sin \alpha)$ at $\alpha = -\pi, \pi, -\frac{\pi}{4}$ and $\frac{3\pi}{4}$ and get the four values 1, 1, $\frac{2}{\sqrt{2}} = \sqrt{2}$ and $-\sqrt{2}$. The maximum rate of change then occurs in the direction $\alpha = -\frac{\pi}{4}$, and $\frac{\partial T}{\partial s} = \frac{\pi\sqrt{2}}{4} e^{-\pi/4}$ *°C/metre* in this direction (assuming we've got the correct units for T, x and y).

Note the temperature variation if x stays fixed at $x = \frac{1}{4}$ (and y increases from y = 0) ... or if y stays fixed at $y = \frac{\pi}{4}$ (and x goes from x = 0 to x = 1)

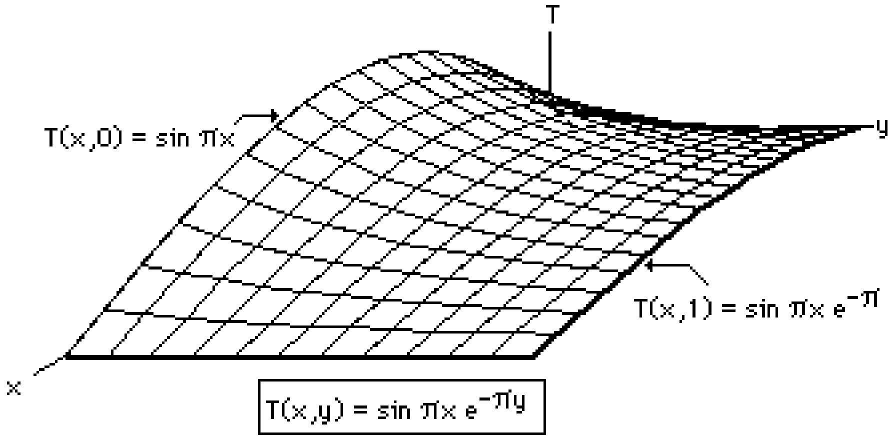


$T(\frac{1}{4}, y) = \sin\frac{\pi}{4}\ e^{-\pi y}$



$T(x, \frac{1}{4}) = \sin \pi x\ e^{-\pi/4}$

Note, too, the surface in 3-D x-y-T space:



LECTURE 16

the GRADIENT

In earlier problems, although we calculated the rate of change of some function z = f(x,y) in a particular direction (first in just the x- and y-directions, then in any α-direction), this leads naturally to the question: "In what direction is $\frac{\partial T}{\partial s}$ a maximum or a minimum?" This is actually of more than just passing interest:

- For a steady distribution of temperature (meaning it doesn't change with time), "heat" (measured, say, in *calories/metre^2/second*) flows is the direction in which the temperature decreases most rapidly.
- The electric field (in an electrostatic environment) is in the direction in which the potential decreases most rapidly.

S: I haven't the foggiest idea of what you're talking about!
P: Does it matter? Just listen to the words ... and be impressed.

- The force of gravity acts in the direction of maximum rate of decrease in gravitational potential. (On the earth, it's downward!).
- The "best" climb up a mountain is in the direction of maximum rate of increase of elevation.
- The "best" skiing, down a mountain is in the direction of maximum rate of decrease of elevation (which is directly opposite the direction of maximum rate of *increase*!).

Now, how do we find this direction?

We digress for just a moment to discuss **vectors** (since they will be a convenient way of identifying a direction ... since vectors are good at pointing).

VECTORS

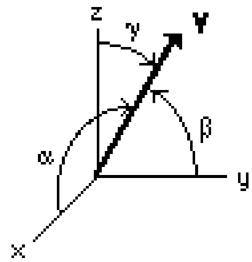
We consider vectors in 3-dimensional space: $\mathbf{V} = [V_1, V_2, V_3]$ (where we'll use square brackets to distinguish vectors from points). We'll denote by $|\mathbf{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2}$ the length of the vector **V** ... and, when convenient, we'll write V as the length, dropping the **bold type**. Every vector has both a length and a direction (... except that we'd have some difficulty associating a direction with the "zero vector" [0,0,0] which has zero length). We will find of particular utility those vectors whose length is "1", called "unit vectors". One reason for this is that such vectors are a convenient way of identifying a particular direction in our 3-D space: just construct a unit vector in the appropriate direction and use it to identify that direction? How else would you indicate a direction?

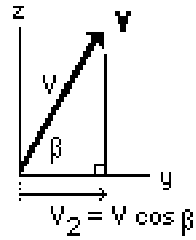
S: How about north-east and south-by-south-west and so on ... or maybe the direction which makes an angle of 30° with such-and-such an axis or maybe ...
P: Okay, there *are* other ways, but you must admit that you *do* use vectors to indicate a direction. Somebody asks "Where?"

and you point. It's not the *length* of your arm that's important, it's the *direction* of your arm. Can I go on?

S: Go ahead.

Suppose we were given a direction as "making angles α, β and γ with the positive x- , y- and z-axes". What would be a vector pointing in this direction? We need a picture ===>>>





To find, say, the y-component of the vector **V** = [V_1, V_2, V_3] ...with length V ... we drop a perpendicular from the tip of **V** to the y-axis and stare at the triangle formed. The side of the triangle which lies along the y-axis is the y-component and it's clearly V cos β. Similarly the x- and z-components are V cos α and V cos γ so we have **V** = [V cos α, V cos β, V cos γ] and if we choose the length V = 1 we get a unit vector in the required direction: $\boxed{\mathbf{u} = [\cos\alpha, \cos\beta, \cos\gamma]}$

S: Wait a minute ... it'll only be a unit vector if its length is "1" and that means $\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$ and maybe I didn't pick my angles that way. I mean, what if I choose angles like $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$ and $\gamma = \frac{\pi}{4}$ then you get **u** = [cos α, cos β, cos γ] = $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and its length is ... uh, $\sqrt{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}$ and, unless I'm mistaken, that's not "1". Right?

P: Right, and you've just discovered one of the mysteries of the universe. There's no such direction as one which makes $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$ and $\gamma = \frac{\pi}{4}$. In fact, if you pick $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$ then you have little choice; γ must $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ or any angle that makes cos γ = 0. See? $\sqrt{\cos^2\alpha + \cos^2\beta} = 1$ so cos γ = 0. These three angles are related. In fact, the three cosines must satisfy

$$\boxed{\text{Direction Cosines satisfy:} \quad \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1}$$

S: Direction cosines?

P: Uh ... sorry, that's what they're called, but you've seen them before. When we were talking about parametric equations for a line in the x-y plane, we said:

$\boxed{\text{The line thru' } (x_0, y_0) \text{ making angles } \alpha, \beta \text{ with the positive x- and y-axes is } x=x_0+s\cos\alpha,\ y=y_0+s\cos\beta}$

See? Direction cosines! In fact, we also noted that, in 2-dimensions, $\alpha + \beta = \frac{\pi}{2}$ so that cos β = sin α and that means that $\cos^2\alpha + \cos^2\beta = \cos^2\beta + \sin^2\alpha = 1$. See? $\boxed{\text{The sum of the squares of the direction cosines is "1"}}$. The vector **u** = [cos α, cos β], in the x-y plane, is a unit vector. In fact, since **V** = [x - x_0, y - y_0] is the vector which goes from (x_0, y_0) to (x,y) then, using the above parametric equations we have **V** = [s cos α, s cos β] where "s" is the distance between (x_0,y_0) and (x,y) hence it's the length of **V** so if s = 1 we'd have a vector of unit length. See how everything hangs together?

S: And comes back to haunt you. I have a question: what's a gradient?

P: Hmmm. Good question. How did you know about gradients?

S: That's the title of this lecture.

$$\boxed{\text{For the function f(x,y), the vector } \textit{grad}\text{ f} = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}] \text{ is called the gradient vector}}$$

The gradient vector (sometimes denoted by ∇ f) has a nice interpretation, but before we say what it is we'll do some examples:

Example: Compute the gradient of the given function at the given point:

(a) $f(x,y) = x^2y$ at (3,7)

(b) $f(x,y) = (x-1)^2+(y-2)^2$ at (0,0)

(c) $f(x,y) = \sin \pi x\, e^{-\pi y}$ at $(\frac{1}{4}, \frac{1}{4})$

(d) $f(x,y) = 100\, e^{-x} \cos y$ at (0,0)

Solution: In each case we use *grad* $f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$:

(a) *grad* $f = [2xy, x^2] = [42, 9]$ at (3,7).

(b) *grad* $f = [2(x-1), 2(y-2)] = [-2, -4]$ at (0,0).

(c) *grad* $f = [\pi \cos \pi x\, e^{-\pi y}, - \pi \sin \pi x\, e^{-\pi y}] = [\frac{\pi}{\sqrt{2}} e^{-\pi/4}, -\frac{\pi}{\sqrt{2}} e^{-\pi/4}]$ at $(\frac{1}{4}, \frac{1}{4})$.

(d) *grad* $f = [- 100\, e^{-x} \cos y, - 100\, e^{-x} \sin y] = [-100,0]$ at (0,0).

All these come from examples considered earlier:

- In (a) we have the rate of change of volume of a square box of side 3 and height 7 when either the side length or height changes ... all wrapped up in a single vector *grad* f = [42,9]!
- In (b) we have the rate of change of elevation of a mountain (if we're sitting at (0,0)) when either x or y changes. Note that this vector, *grad* f = [-2, -4], points in a direction with slope $\frac{-4}{-2} = 2$ and that's the direction (as we saw earlier) which gives the maximum rate of increase of elevation. Is that an accident?
- In (c) we have the rate of change of temperature in a plate, at the point $(\frac{1}{4}, \frac{1}{4})$, and we note that *grad* f = $[\frac{\pi}{\sqrt{2}} e^{-\pi/4}, -\frac{\pi}{\sqrt{2}} e^{-\pi/4}]$ points "south-east", the direction making an angle $\alpha = -\frac{\pi}{4}$ with the positive x-direction and that's the direction (as we saw earlier) which gives the maximum rate of increase of temperature. Is that an accident?
- In (d) we have the rate of change of temperature at (0,0) and we note that *grad* f = [-100,0] points due west (in the *negative* x-direction) whereas the temperature <u>decreases</u> most rapidly in the *positive* x-direction. Is that an accident?

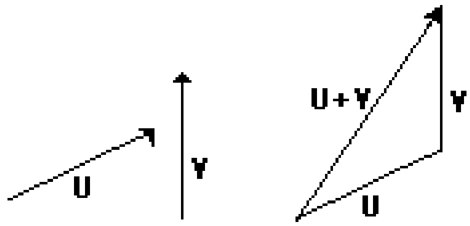
As you might expect, these are not accidents: *grad* $f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$, when evaluated at some point x = a, y = b, gives a vector which points in the direction in which f(x,y) increases most rapidly! (Or, to put it differently, *grad* f points <u>opposite</u> to the direction in which f(x,y) <u>decreases</u> most rapidly ... as in (d), above.)

S: Hold on! Is that some kind of proof? I mean, do a couple of examples prove anything? I mean ...

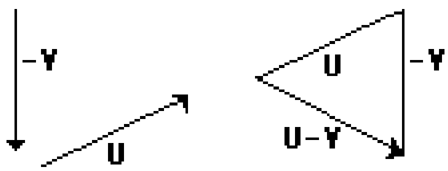
P: No, of course not, but I can prove it. Pay attention. I have to talk a little more about vectors:

A little More About Vectors

Let's talk about 2-D vectors, $\mathbf{U} = [U_1, U_2]$ and $\mathbf{V} = [V_1, V_2]$ etc. Every such vector has both a length and a direction. The length of **U**, denoted by | U |, is $\sqrt{U_1^2 + U_2^2}$. To get the direction of **U** we could take the ratio $\frac{\text{y-component}}{\text{x-component}} = \frac{U_2}{U_1}$ and that would give us the slope of this vector.



We can also **add** vectors: The SUM of these two vectors is **U** + **V** = $[U_1+V_1, U_2+V_2]$. Geometrically, we SUM them by sliding **V** (while maintaining its length and direction!) so its "tail" touches the "head" of **U**. This is *vector addition* and the SUM is the vector which goes from the "tail" of **U** to the "head" of **V**. To get its length requires the cosine law (assuming we know the lengths and directions of **U** and **V** so we know two sides and the contained angle for the triangle whose third side is the length of **U** + **V**).



We can also **subtract** vectors: the DIFFERENCE is **U** - **V** = $[U_1-V_1, U_2-V_2]$. Geometrically, we SUBTRACT them (1) changing the direction of **V** (while maintaining its length!), and that gives - **V**, then (2) finding the SUM of **U** and (- **V**) ... thereby reducing the problem to one we've already considered!.

There is a natural question to ask: *Can we multiply vectors?* The answer is, of course, yes.

S: Of course? Why "of course"? It's not obvious how ...
P: We can define what we mean by "muliplying" **U** and **V** in any way we wish. I could, for example, say that the PRODUCT **U V** is the vector $[U_1V_2, U_2V_1]$ ... i.e. [1st component times 2nd, 2nd component times 1st]. Why not? You can invent a product yourself if you'd like. Who can argue? It's a definition, right?
S: Is *that* the definition of the product?
P: It's not the one I want to talk about. It does, however, have the nice property that if I multiply any vector by the zero vector, [0,0], the result is the zero vector. However, if I want to add **U** and **V** then multiply by **W** = $[W_1, W_2]$ and I write this as **W** (**U** + **V**) which is **W** multiplied by $[U_1+V_1, U_2+V_2]$ and I use my invented "multiplication rule", I'd get **W** (**U** + **V**) = $[W_1(U_2+V_2), W_2(U_1+V_1)]$. On the other hand, if I multiplied **U** + **V** by **W**, writing it as (**U** + **V**) **W**, I'd get (again using my "invention"): $[(U_1+V_1)W_2, (U_2+V_2) W_1]$. They're not the same ... but that's okay. What is most bothersome about this invented "multiplication" is that ... can you see what's bothersome?
S: It doesn't bother me.
P: I have no idea how to multiply vectors in 3-dimensional space. You see, this recipe doesn't generalize easily. And there are other things I don't like about it. I have no idea if it's useful, and ...
S: Why don't you just tell me what you <u>do</u> like.

the DOT Product

The DOT product between two vectors **U** = $[U_1, U_2]$ and **V** = $[V_1, V_2]$ is defined by $U \bullet V = U_1V_1 + U_2V_2$.
Let's make a fuss about this:

$$\boxed{\mathbf{U}\bullet\mathbf{V} = U_1V_1 + U_2V_2 \text{ is the DOT product between vectors } \mathbf{U} = [U_1, U_2] \text{ and } \mathbf{V} = [V_1, V_2]}$$

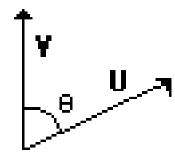
Example: Calculate the dot product between the following vectors:
(a) **U** = [2, -1] and **V** = [0, 5]
(b) **U** = [2,5,-1] and **W** = [5,0,1]
(c) **P** = [5,1,1] and **Q** = [-1,4,1]
Solution:
(a) **U•V** = [2, -1]•[0, 5] = (2)(0) + (-1)(5) = -5
(b) **U•W** = [2,5,-1]•[-5,0,1] = (2)(-5) + (5)(0) + (-1)(1) = -11
(c) **P•Q** = [5,1,1]•[-1,4,1] = (5)(-1) + (1)(4) + (1)(1) = 0

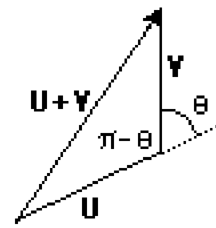
S: But the dot product isn't even a vector, right? Doesn't it have to be a vector? I mean ...
P: You're right, it's NOT a vector ... it's a SCALAR (i.e. a number) and for that reason it's sometimes called the SCALAR

PRODUCT. And since it's a definition we don't have to justify it ... except we'd look pretty foolish if it were useless. Another thing: notice how it generalizes so nicely to 3-D vectors (as in (c), above)? Just multiply the components, pair-wise, and add up all these products.

S: Why call it the DOT product?

P: Didn't you see the big DOT? Anyway, now I have to show it's important in the sense that this "product" actually occurs in meaningful problems. In particular I want to answer the question: "Does the gradient vector *grad* f(x,y) give the direction in which f(x,y) increases most rapidly?"





Earlier we mentioned that the length of the sum vector **U**+**V** could be obtained as the length of a side of a triangle, using the cosine law. We'll do that now and something wonderful will happen:

Suppose θ is the angle between **U** and **V**, as shown. The length of the sum vector **U**+**V** is given by $|\mathbf{U}+\mathbf{V}|^2 = |\mathbf{U}|^2 + |\mathbf{V}|^2 - 2\,|\mathbf{U}|\,|\mathbf{V}| \cos\theta$. Now we insert the lengths of **U**, **V** and **U**+**V** and get:
$(U_1+V_1)^2 + (U_2+V_2)^2 = (U_1^2+U_2^2) + (V_1^2+V_2^2) - 2\,|\mathbf{U}|\,|\mathbf{V}| \cos\theta$, then square the terms on the left-side and cancel and get, finally: $\boxed{U_1V_1 + U_2V_2 = |\mathbf{U}|\,|\mathbf{V}| \cos\theta}$. This is quite remarkable. The left-side is just the DOT product, so we have:

$$\mathbf{U}\bullet\mathbf{V} = U_1V_1 + U_2V_2 = |\,\mathbf{U}\,|\,|\,\mathbf{V}\,| \cos\theta$$

where θ is the angle between vectors **U**= $[U_1, U_2]$ and **V** = $[V_1, V_2]$

In words, the DOT product of two vectors is a number which can be obtained *either* by multiplying their components, pair-wise, and adding OR by multiplying the product of their lengths by the cosine of the angle between them. If you know U_1, U_2 and V_1, V_2 then you can compute their length hence the angle between these two vectors.

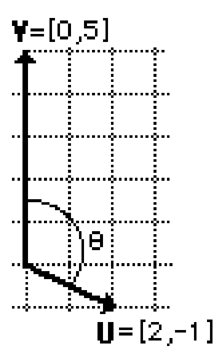
Example: Calculate the angle between the following vectors:

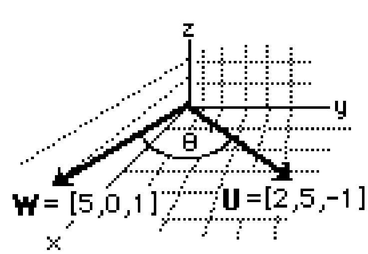
(a) **U** = [2, -1] and **V** = [0, 5]

(b) **U** = [2,5,-1] and **W** = [-5,0,1]

(c) **P** = [5,1,1] and **Q** = [-1,4,1]

Solution:





(a) $\mathbf{U}\bullet\mathbf{V} = -5$ and $|\mathbf{U}| = \sqrt{2^2+1^2} = \sqrt{5}$ and $|\mathbf{V}| = \sqrt{0^2+5^2} = 5$ and $\mathbf{U}\bullet\mathbf{V} = |\mathbf{U}|\,|\mathbf{V}| \cos\theta$ so we have $\cos\theta = -\frac{1}{\sqrt{5}}$ hence $\theta = \arccos\left(-\frac{1}{\sqrt{5}}\right)$ where we choose the angle in $0 \le \theta \le \pi$. In fact, since $\cos\theta \le 0$, then $\frac{\pi}{2} \le \theta \le \pi$.

(b) $\mathbf{U}\bullet\mathbf{W} = -11$ and $|\mathbf{U}| = \sqrt{2^2+5^2+1^2} = \sqrt{30}$ and $|\mathbf{W}| = \sqrt{5^2+0^2+1^2} = \sqrt{26}$ and $\mathbf{U}\bullet\mathbf{W} = |\mathbf{U}|\,|\mathbf{W}| \cos\theta$ so we have $\cos\theta = -\frac{11}{\sqrt{30}\sqrt{26}}$ hence $\theta = \arccos\left(-\frac{11}{\sqrt{30}\sqrt{26}}\right)$ where, since $\cos\theta \le 0$, then $\frac{\pi}{2} \le \theta \le \pi$.

(c) $\mathbf{P}\bullet\mathbf{Q} = [5,1,1]\bullet[-1,4,1] = (5)(-1) + (1)(4) + (1)(1) = 0$ and since $\mathbf{P}\bullet\mathbf{Q} = |\mathbf{P}|\,|\mathbf{Q}| \cos\theta$ where θ is the angle between **P** and **Q**, then $\cos\theta$ must be zero, so $\theta = \frac{\pi}{2}$ and we conclude that **P** and **Q** are perpendicular to each other!

S: Is that a way to tell if two vectors are perpendicular?
P: Yes. If $\mathbf{P}\bullet\mathbf{Q} = 0$ then they must be perpendicular.
S: But what if $\mathbf{P} = [0, 0]$?
P: Oh ... yes, that's ... uh, you're quite right. In that case $\mathbf{P}\bullet\mathbf{Q} = 0$ too. Well, if neither **P** nor **Q** has zero length (i.e. $|\mathbf{P}| \ne 0$ and $|\mathbf{Q}| \ne 0$) then they'll be perpendicular if $\mathbf{P}\bullet\mathbf{Q} = 0$.

Whenever we see an expression like A B + C D we can imagine two vectors [A, C] and [B, D] and we can interpret A B + C D as their DOT product and even find the angle between them.

S: Is *that* useful?
P: Pay attention:

The directional derivative (i.e the rate of change of f(x,y) in the direction α) is $\frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha$ and this has the form A B + C D so we can write it as a DOT product between two vectors: $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}] \bullet [\cos\alpha, \sin\alpha]$. We recognize the first of these vectors as none other than *grad* f. And the second? It's just a unit vector in the α-direction!

> *grad* f • **u** is the directional derivative in the direction of the unit vector **u**

P: What does that say to you, about the direction of maximum increase of f?
S: Huh?
P: Can't you see? You're standing at some place and you calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at that place. That gives a vector, *grad* f = $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$ and this vector points in some direction. Now you want to determine how quickly f changes in some *other* direction, namely the direction in which $\mathbf{u} = [\cos\alpha, \sin\alpha]$ points. You just take the DOT product *grad* f • **u** and that gives you the rate of change in the α-direction. So now you vary this α-direction (i.e. the direction of the unit vector **u**), computing *grad* f • **u** for each new α-direction. Remember that *grad* f = $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$ isn't changing. It just points in some fixed direction, waiting patiently for you to rotate **u** and compute *grad* f • **u** again and again until you've found the largest rate of change (i.e. the largest value of *grad* f • **u**). So? In what direction should you move to achieve this greatest rate of change in f?
S: I'd say you should move in the direction of *grad* f?
P: Excellent! Why do you say that?
S: Well ... it's just sitting there, pointing ... and besides, it's the only direction I can think of.

We've been able to generate various expressions for the rate of change of f in a given direction. It's the most recent expression that answers the question: *"Does the gradient vector grad f give the direction in which f increases most rapidly?"*. We string out these expressions and look carefully at the last. It says something exciting:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}] \bullet [\cos\alpha, \sin\alpha]$$
$$= \textit{grad}\ f \bullet \mathbf{u} = |\,\textit{grad}\ f\,|\,|\,\mathbf{u}\,|\cos\theta = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ \cos\theta$$

Clearly we obtain the maximum value of $\frac{\partial f}{\partial s}$ by choosing the maximum value of $\cos\theta$ which means we should move in the direction $\theta = 0$ which means the angle between **u** and *grad* f should be zero which means **u** should be in the direction of *grad* f.

S: Hold on! To get a larger rate of change you just make $\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$ bigger, right?

P: Wrong. You're standing at some place and you evaluate *grad* f at that place, meaning you calculate the partial derivatives at that place, and they're just numbers and they give you a fixed gradient vector which points in a fixed direction. I said that already! NOW you vary **u** and that means you're varying <u>only</u> θ which means the rate of change is greatest when ...

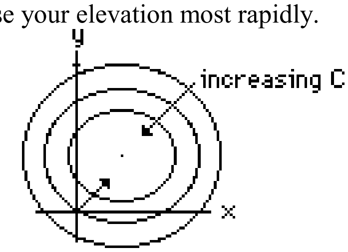
S: Okay, okay ... I get it. And I assume that once you move a little ways in the direction of *grad* f then you have to do this all over again, I mean compute another *grad* f and then move in <u>that</u> direction and so on and so on.

P: Yes, if you insist upon moving in the direction where f increases most rapidly.

S: But we've done that before, right? In sounds familiar. We were on the side of a mountain as I recall ...

Example: You are standing on the side of a mountain whose elevation is given by $z = 95 - x^2 - y^2 + 2x + 4y$ metres, where x = 0, y = 0 is your location, so z = 95 is your elevation. Sketch the level curves in your neighbourhood and determine in what direction you should climb so as to increase your elevation most rapidly.

Solution:



The direction of maximum increase of $z = 95 - x^2 - y^2 + 2x + 4y$ is just the direction of the vector *grad* z = $[\frac{\partial z}{\partial x}, \frac{\partial f}{\partial y}] = [-2x+2, -2y+4]$ which, at (0,0), is [2,4]. The "slope" of this vector is $\frac{4}{2} = 2$, just as we obtained earlier when we took the direction to be perpendicular to the LEVEL CURVE through (0,0).

Earlier, we did another problem where the LEVEL CURVES were given by $(x-1)^2 + 2(y-2)^2 = 100 - C$ (so they were ellipses rather than circles so it wasn't so obvious what direction gave the maximum increase in elevation). Nevertheless, this magic direction is the still the direction of *grad* f where $f(x,y) = 100 - (x-1)^2 - 2(y-2)^2$. Actually, we could take $f(x,y) = -x^2 + 2x - 2y^2 + 8y$ or $f(x,y) = -x^2 + 2x - 2y^2 + 8y + 15$ etc. since adding a constant to f(x,y) doesn't change its partial derivatives and *grad* f depends only upon these partials! Anyway, *grad* f = [-2x+2,-4y+8] = [2,8] at (0,0) and this gives the direction of maximum increase: the "slope" of *grad* f is $\frac{8}{2} = 4$ (as we obtained earlier when we took the direction to be perpendicular to the LEVEL CURVES).

S: Are you saying that *grad* f is *always* perpendicular to the level curves?

P: You got it. At every point P(a,b) the gradient vector $[\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)]$ is perpendicular to that level curve, f(x,y) = *constant*, which passes through P, namely f(x,y) = f(a,b). Let's make a note of that:

The gradient vector *grad* f = $\left[\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right]$ at a point P(a,b) is perpendicular to f(x,y) = f(a,b), the level curve through P

LECTURE 17

more on the GRADIENT, and the CHAIN RULES

Let's recap:

- Given a function of two variables, f(x,y), the LEVEL CURVES are f(x,y) = C where C is a constant.
- The level curve through a given point P(a,b) has C = f(a,b), so is f(x,y) = f(a,b).
- The rate of change of f(x,y) in the x-direction is $\frac{\partial f}{\partial x}(a,b)$, and $\frac{\partial f}{\partial y}(a,b)$ is the rate of change in the y-direction.
- At each point P(a,b) the gradient vector is defined as: $\nabla f = \textit{grad}\, f(a,b) = \left[\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right]$
- The rate of change of f(x,y) in the direction of the unit vector **u** = [cos α, sin α], namely the directional derivative (denoted by $\frac{\partial f}{\partial s}$), is given by either *grad* f (a,b) • **u** = $\frac{\partial f}{\partial x}(a,b)\cos\alpha + \frac{\partial f}{\partial y}(a,b)\sin\alpha$ or, equivalently, by $\sqrt{\left(\frac{\partial f}{\partial x}(a,b)\right)^2+\left(\frac{\partial f}{\partial y}(a,b)\right)^2}\cos\theta$, where θ is the angle between *grad* f(a,b) and **u**.
- The direction of *grad* f(a,b) is perpendicular to the level curve through P(a,b) (i.e perpendicular to f(x,y) = f(a,b)).
- The maximum rate of change of f(x,y), at the point P(a,b), is in the direction of *grad* f(a,b) (since this maximum rate of change occurs when cos θ = 1, or θ = 0, meaning **u** points in the direction of *grad* f(a,b)).
- The magnitude of this maximum rate of change is $\sqrt{\left(\frac{\partial f}{\partial x}(a,b)\right)^2+\left(\frac{\partial f}{\partial y}(a,b)\right)^2}$.

S: Huh? Where did that come from? I mean, we never talked about what the maximum rate *was* ... just how to move to achieve it.

P: Since $\frac{\partial f}{\partial s} = \sqrt{\left(\frac{\partial f}{\partial x}(a,b)\right)^2+\left(\frac{\partial f}{\partial y}(a,b)\right)^2}\cos\theta$ and the maximum occurs when θ = 0, guess what this maximum is?

S: So *grad* f not only gives the direction but also the size of this maximum increase, right?

P: Right! Since *grad* f is a vector it has both length and direction. The length provides the maximum rate of change and the direction provides the direction in which this maximum increase takes place. Nice, eh?

S: Mamma mia! You sure get your money's worth from good ol' *grad* f, right? When we were standing on the side of the mountain, $z = 95 - x^2 - y^2 + 2x + 4y$ and *grad* z = [-2x+2, -2y+4] = [2,4] at P(0,0) so the maximum rate of increase was $\sqrt{2^2+4^2} = \sqrt{20}$, right?

P: In *metres/metre*.

S: And when the level curves were ellipses, like $f(x,y) = 100 - (x-1)^2 - 2(y-2)^2$ = constant, then *grad* f = [2,8] at P(0,0) so the maximum change was $\sqrt{(2)^2+(8)^2} = \sqrt{68}$.

P: The maximum <u>rate</u> of change was $\sqrt{68}$ *metres/metre*. Right.

S: I think the gradient vector is probably pretty useful, right?

- If T(x,y,z) is the temperature at P(x,y,z) where x, y and z might be the latitude, longitude and elevation of the point P, the heat flows in the direction of maximum decrease in T(x,y,z), namely in the direction of - *grad* T. In fact, if **H** is the heat flow vector (measured in, say, *calories/metres*^{2}*/second*) then **H** = - k *grad* T where k depends upon the conductivity of the medium in which the heat is flowing.
- If V(x,y,z) is the voltage potential at P, then **E**, the electric field, is in the direction - *grad* V. In fact, **E** = - *grad* V.
- If U(x,y,z) is the gravitational potential at P(z,y,z), then **F**, the force of gravity acts in the direction - *grad* V. In fact, **F** = - *grad V*.

S: I don't know anything about that potential stuff ...

P: It doesn't matter. Just be impressed with the usefulness of the gradient vector.

Example: The distribution of a certain type of plant is given by its density: $P(x,y) = 100 \sin^2 x\, e^{-3y}$ plants/km^2, where x and y measure distance, in kilometres, from some origin. Compute:

(a) the rate of change of P(x,y) at x = 2, y = 1, in both the x- and y-directions,

(b) the magnitude of the maximum rate of increase of plants at (2,1), and
(c) the direction in which this maximum increase occurs.

Solution:

(a) *grad* P = [$\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$] = [200 sin x cos x e^{-3y}, -300 $\sin^2 x$ e^{-3y}] = [200 sin 2 cos 2 e^{-3}, -300 $\sin^2 2$ e^{-3}] at (2,1). We have $\frac{\partial P}{\partial x}$ = 200 sin 2 cos 2 e^{-3} and $\frac{\partial P}{\partial y}$ = -300 $\sin^2 2$ e^{-3} at (2,1).

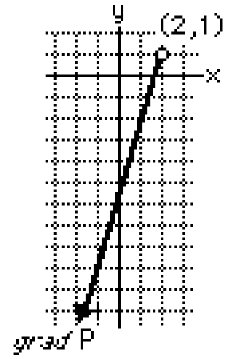
(b) The length of *grad*P is $\sqrt{(200 \sin 2 \cos 2\ e^{-3})^2 + (-300 \sin^2 2\ e^{-3})^2}$ and this is the maximum rate of increase (measured in *plants/km²/km*).

(c) The direction of *grad* P gives the direction in which this maximum increase takes place.

S: But what are those numbers ... and that angle?

P: I'll let MAPLE do it MAPLE. I'll define P, then compute $P_x = \frac{\partial P}{\partial x}$ and $P_y = \frac{\partial P}{\partial y}$ using the MAPLE commands **diff(P,x)** and **diff(P,y)**, then I'll **subs**titute x = 2 and y = 1 and **eval**uate as a **f**loating (decimal) number (using the **subs** and **evalf** commands), then I'll take the length of the gradient vector as $\sqrt{P_x{}^2 + P_y{}^2}$ (and uses **sqrt** to denote the square root) and that'll give me the maximum rate of change, then I'll compute $\frac{P_y}{P_x}$ and that'll give me the tangent of the angle, then I'll use the arctan function to get the angle, but MAPLE is pretty smart and knows that the range of arctan is $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ so it'll give me the angle in *that* range but I notice that my gradient points into the third quadrant so I'll just add π (which MAPLE calls **Pi**) and I'll ask MAPLE to **evalf** this angle:

```
• P:=100*sin(x)^2*exp(-3*y);
                                   2
                     P := 100 sin(x)  exp(- 3 y)

• Px:=diff(P,x);
                 Px := 200 sin(x) exp(- 3 y) cos(x)

• Py:=diff(P,y);
                                      2
                   Py := - 300 sin(x)  exp(- 3 y)

• Px:=evalf(subs(x=2,y=1,Px));
                         Px := -3.767897758

• Py:=evalf(subs(x=2,y=1,Py));
                         Py := -12.34951020

• MaxRate:=sqrt(Px^2+Py^2);
                       MaxRate := 12.91152414

• TanTheta:= Py/Px;
                       TanTheta := 3.277559794

• Theta:=arctan(TanTheta);
                         Theta := 1.274662618

• Theta:=evalf(Theta+Pi);
                         Theta := 4.416255272
```



S: That angle isn't in the 3rd quadrant, it's ...

P: That angle IS in the 3rd quadrant. Remember, it's in RADIANS. In degrees it'd be about $4.4\left(\frac{180}{\pi}\right)$ or about 252˚.

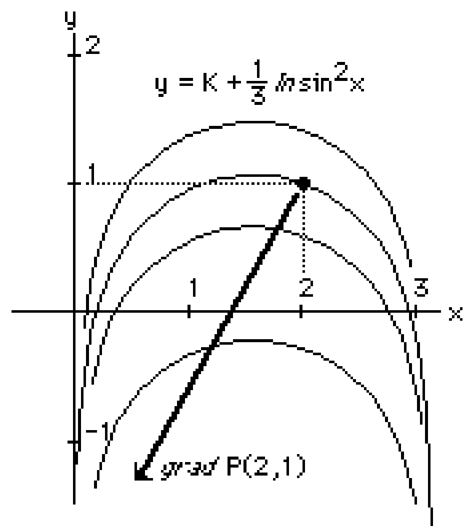
S: How did Maple know to give it to you in radians?

P: Maple has a PhD in mathematics.

S: Could you plot a few level curves. I'd like to see that *grad* P really <u>is</u> perpendicular.

P: Be glad to. I'll have to plot $100 \sin^2 x\, e^{-3y} = C$ for various values of the constant C and that means $e^{3y} = \frac{100}{C} \sin^2 x$ and that means

$3y = \ln\left(\frac{100}{C}\sin^2 x\right) = \ln\frac{100}{C} + \ln \sin^2 x$ and that means I can plot $y = K + \frac{1}{3} \ln \sin^2 x$ for various values of the constant K. I'd better avoid places where sin x = 0, of course (because of the logarithm), but we want the level curve which passes through x = 2, y = 1 and that means choosing $1 = K + \frac{1}{3} \ln \sin^2 2$ so we'll just choose K-values near this. Here's a plot of a few level curves. Notice that the gradient vector at (2,1) is indeed perpendicular to the level curve through (2,1).



S: And look at how the level curves head off to infinity as x approaches 0 or π ... that's where the sin x = 0 so the *ln* blows up, right?

P: Yes indeed.

S: But weren't we talking about plants? Are you saying that the density of plants becomes infinite as x approaches π kilometres? Is that reasonable?

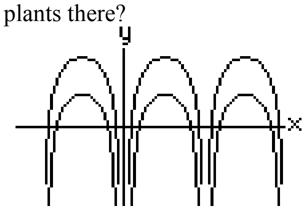
P: No, no. Everywhere along the level curve through (2,1) there is a constant density, namely P(2,1). In fact the equation of this level curve is P(x,y) = P(2,1) or $100 \sin^2 x\, e^{-3y} = 100 \sin^2 2\, e^{-3} \approx 4.1$ so that's the density and what the graph shows is that if you want to maintain the same constant density of 4.1 *plants/metres*^{2} you'd have to follow this level curve and that means heading due south as you approach x = π kilometres.

S: So nothing is really infinite, right? But it sure looks that way, right?

P: If that level curve were a picture of a highway taken from an airplane, would you see any infinities? No.

S: One other thing. What happens beyond x = π or maybe x < 0. Are there no plants there?

P: You figure it out.



S: Well ... I'd look at the level curves $100 \sin^2 x\, e^{-3y} = C$ or, as you put it, $y = K + \frac{1}{3} \ln \sin^2 x$, and I'd see that ... uh ... $\sin^2 x$ repeats itself so the curve repeats itself so you've just plotted the level curves in 0 < x < π but there would be others and they'd look the same except they'd be shifted right or left by π kilometres so they'd look something like this  ... how'm I doin' boss?

P: Have you taken this course before?

The CHAIN RULES

It is tempting to repeat <u>everything</u> that we did for functions f(x) of a single variable, generalizing to functions of two (or more) variables. Although many of these recipes generalize quite easily (... we've already seen parametric equations of a line in 3-D, and first and second derivatives, and we've even managed to get derivatives in a general direction which was a problem which didn't even arise in single-variable calculus!).

Now it's time to generalize the Chain Rule!

Alas, it's not clear what we should consider. Should we consider z = f(x,y) and each of x and y are functions of <u>two</u> *other* variables? Or maybe z = f(x,y) and each of x and y is a function of just a <u>single</u> *other* variable. Or perhaps u = f(z) where z is a function of <u>two</u> variables, such as z = f(x,y) !?^%$*

Let's do the last, first:

Example: If $u = \sin z$ and $z = x^2 + y^2$ (so that u is *indirectly* a function of two variables x and y), then compute $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Solution: We can clearly just substitute for z and get $u = \sin(x^2+y^2)$, hence $\frac{\partial u}{\partial x} = \cos(x^2+y^2)\, 2x$ and $\frac{\partial u}{\partial y} = \cos(x^2+y^2)\, 2y$ ... and we're finished. However, we'd like to identify the "Rule", so we notice that the factor $\cos(x^2+y^2)$ is just cos z which is just $\frac{du}{dz}$ and the two pieces 2x and 2y are just $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively. This means $\frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x}$ and $\frac{\partial u}{\partial y} = \frac{du}{dz}\frac{\partial z}{\partial y}$ and <u>that's</u> the "Rule". Notice that we needn't write $\frac{\partial u}{\partial z}$ because u is a function of the <u>single</u> variable z and we need only differentiate using our familiar single-variable calculus.

We make a fuss about this new Chain Rule:

$$\text{If } u = f(z) \text{ and } z = z(x,y) \text{ then } \frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{du}{dz}\frac{\partial z}{\partial y}$$

Notice how the terms are formed. If u is measured in degrees, z in metres, x in kilograms and y in seconds, then $\frac{\partial u}{\partial x}$ is measured in *degress/kg* and so is $\frac{du}{dz}\frac{\partial z}{\partial x}$ because it's (*degrees/metre*) (*metres/kg*). That observation alone is almost enough to generate this Chain Rule. How else would you combine the various derivatives $\frac{du}{dz}$ *degrees/metre*, $\frac{\partial z}{\partial x}$ *metres/kg* and $\frac{\partial z}{\partial y}$ *metres/second* to form *degrees/kg* ?

Another observation: derivatives (even partial derivatives) are "rates of change" and indicate how rapidly one variable changes compared to another: $\frac{du}{dz}$ compares the change in u with the change in z whereas $\frac{\partial z}{\partial x}$ compares the change in z with the change in x. Hence, if $\frac{du}{dz} = 6$ it means that u changes 6 times more rapidly than z, while $\frac{\partial z}{\partial x} = 7$ means z changes 7 times more rapidly than x. So what's $\frac{\partial u}{\partial x}$, which tells how rapidly u changes compared to changes in x? Obviously it's (6)(7) = 42.

The Canadian humorist Stephen Leacock knew all about the chain rule when he wrote of A working twice as hard as B who worked twice as hard as C.

This Chain Rule: common sense couched in mathematical jargon.

Example: Compute:

(a) $\frac{\partial u}{\partial x}$ if $u = e^z$ and $z = \sin x\, e^{-2y}$

(b) $\frac{\partial T}{\partial y}$ if $T = u^2$ and $u = \mathit{ln}\, x\, e^{x+y}$

(c) $\frac{\partial V}{\partial T}$ if $V = \tan w$ and $w = T^2V + \cos(VT)$

(d) $\frac{\partial P}{\partial z}$ if $P = e^{u^2}$ and $u = x^2 + y^2z^2 + z \sin t$

Solution:

(a) $\frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x} = e^z\left(\cos x\, e^{-2y}\right)$ (b) $\frac{\partial T}{\partial y} = \frac{dT}{du}\frac{\partial u}{\partial y} = 2u\left(\mathit{ln}\, x\, e^{x+y}\right)$

(c) $\dfrac{\partial V}{\partial T} = \dfrac{dV}{dw}\dfrac{\partial w}{\partial T} = \sec^2 w\,\big(2TV - \sin(VT)\,V\big)$

(d) $\dfrac{\partial P}{\partial z} = \dfrac{dP}{du}\dfrac{\partial u}{\partial z} = 2u\,e^{u^2}\,(\,2y^2 z + \sin t\big)$

S: That's pretty easy ... I think, but does one example <u>prove</u> this new chain rule?
P: Pay attention. I'll go through it, but with different variable names:

Suppose that P = f(u) and u depends upon a host of <u>other</u> variables, one of which is z. Let z change by an amount Δz so that u will change by some amount, call it Δu (which we'll assume isn't zero because we're going to divide by Δu !), and that'll make P = f(u) change by some amount, call it ΔP. The partial derivative we seek is $\lim\limits_{\Delta z \emptyset 0} \dfrac{\Delta P}{\Delta z}$ and we simply write $\dfrac{\Delta P}{\Delta z} = \dfrac{\Delta P}{\Delta u}\dfrac{\Delta u}{\Delta z}$ then take the limit as $\Delta z \to 0$ (and that'll make $\Delta u \to 0$ as well) so we get, in the limit, $\dfrac{dP}{du}\dfrac{\partial u}{\partial z}$ where we use "d" when we're differentiating a function of a single variable and "∂" when we're differentiating a function of many variables. That gives us our Chain Rule.

Of course we've been considering the case where P depends upon a <u>single</u> variable u which in turn depends upon <u>many</u> others. It's like saying the pressure P of a bottle of gas depends upon its temperature u (and nothing else), but the temperature u depends upon the location of the bottle ... for example, x = latitude, y = longitude and z = elevation, because these three variables will determine the temperature. In asking for $\dfrac{\partial P}{\partial z}$ we are asking for the rate of change of pressure (within the bottle) as the elevation increases (assuming a change in elevation will change the temperature, u).

S: You began with the title "Chain Rules". That's plural. I assume ...
P: Yes, there are others ... lots of them.

Suppose, now, that P depends upon <u>many</u> variables and each of these depends upon a <u>single</u> variable. In a sense this is the opposite of what we've just considered. We write P = f(x,y,z) where x = x(t), y = y(t), z = z(t). It's like saying the pressure of our bottle of gas depends upon its location (x,y,z), but we're in an airplane so the location is changing with time, t. In fact, x = x(t), y = y(t), z = z(t) are parametric equations for the path of the plane!

Now how does the pressure P change with time? We want $\dfrac{dP}{dt}$.

We change time from t to t + Δt, causing a change in position from (x,y,z) to ($x+\Delta x, y+\Delta y, z+\Delta z$), hence a change in pressure from P(x,y,z) to $P(x+\Delta x,y+\Delta y,z+\Delta z)$: call this change ΔP. What we want is $\lim\limits_{\Delta t \emptyset 0} \dfrac{\Delta P}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{P(x+\Delta x,y+\Delta y,z+\Delta z) - P(x,y,z)}{\Delta t}$. There's a trick here and we've done it before. Remember?

S: Nope.
P: When we have too many things changing at once, like x <u>and</u> y <u>and</u> z?
S: Nope, my mind is a complete blank.
P: Remember when we wanted to find the rate of change of f(x,y) is some direction and both x and y were changing? What did we do?
S: Aah, yes ... we changed them one-at-a-time. Go ahead, do it!

We write the total change in P, namely $\Delta P = P(x+\Delta x,y+\Delta y,z+\Delta z) - P(x,y,z)$, as the sum of changes when each of x, y and z change one-at-a-time, and we'll call the individual changes ΔP_x, ΔP_y and ΔP_z:

$\Delta P_x = P(x+\Delta x,y,z) - P(x,y,z)$ is the change in P when only x changes, and we move from (x,y,z) to $(x+\Delta x,y,z)$.

$\Delta P_y = P(x+\Delta x,y+\Delta y,z) - P(x+\Delta x,y,z)$ is the change when we move from $(x+\Delta x,y,z)$ to $(x+\Delta x,y+\Delta y,z)$.

$\Delta P_z = P(x+\Delta x,y+\Delta y,z+\Delta z) - P(x+\Delta x,y+\Delta y,z)$, when we move from $(x+\Delta x,y+\Delta y,z)$ to $(x+\Delta x,y+\Delta y,z+\Delta z)$.

Then $\Delta P = P(x+\Delta x,y+\Delta y,z+\Delta z) - P(x,y,z) = \Delta P_x + \Delta P_y + \Delta P_z$ hence $\frac{\Delta P}{\Delta t} = \frac{\Delta P_x}{\Delta t} + \frac{\Delta P_y}{\Delta t} + \frac{\Delta P_z}{\Delta t}$ and we need only compute each of these three limits, as $\Delta t \to 0$, which shouldn't be so bad because only one variable is changing!

We'll start with $\lim\limits_{\Delta t \to 0} \frac{\Delta P_x}{\Delta t}$.

Note that $\frac{\Delta P_x}{\Delta t} = \frac{P(x+\Delta x,y,z) - P(x,y,z)}{\Delta t} = \frac{P(x+\Delta x,y,z) - P(x,y,z)}{\Delta x} \frac{\Delta x}{\Delta t}$ where again we used the same trick as we used when considering directional derivatives, because now each of $\frac{P(x+\Delta x,y,z) - P(x,y,z)}{\Delta x}$ and $\frac{\Delta x}{\Delta t}$ has a recognizable limit.

We get $\lim\limits_{\Delta t \to 0} \frac{\Delta P_x}{\Delta t} = \frac{\partial P}{\partial x}\frac{dx}{dt}$ (where we use "d" in $\frac{dx}{dt}$ because x depends upon a single variable, t) .

In a similar manner we can compute the limits of $\lim\limits_{\Delta t \to 0} \frac{\Delta P_y}{\Delta t}$ and $\lim\limits_{\Delta t \to 0} \frac{\Delta P_z}{\Delta t}$.

The sum of the three limits we'll call $\frac{dP}{dt}$ rather than $\frac{\partial P}{\partial t}$ because there is only a single independent variable which all others depend upon:

$$\boxed{\text{If } P = f(x,y,z) \text{ and } x = x(t),\ y = y(t) \text{ and } z = z(t), \text{ then } \frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt}}$$

Although you may be tired of this dimensional stuff, I'll repeat it anyway:

If P is measured in *hectares* and x, y, z in *kilograms*, *metres* and *seconds* and t in *years*, then $\frac{dP}{dt}$ is measured in *hectares/year* as are each of the three terms. For example, $\frac{\partial P}{\partial y}\frac{dy}{dt}$ = (*hectares/metre*) (*metres/year*).

Example: The air pressure at a point (x,y,z) is given by $P(x,y,z) = K\,(3-2\sin x)\,(1+\cos^2 y)\,e^{-z}$ *pascals*, where K is some constant and x, y and z are measured in *kilometres*. If the point moves along a curve descibed by $x = t^2$, $y = \ln(1+t)$, $z = t$, where t is the time in *hours*, then compute $\frac{dP}{dt}$ at the time t = 2 hours.

Solution: When t = 2, we are at the point: x = 4, y = *ln* 3 and z = 2 and moving so that $\frac{dx}{dt} = 2t = 4$, $\frac{dy}{dt} = \frac{1}{t} = .5$ and $\frac{dz}{dt} = 1$ *km/hour*.

Further, $\frac{\partial P}{\partial x} = K\,(-2\cos x)\,(1+\cos^2 y)\,e^{-z} = K\,(-2\cos 4)\,(1+\cos^2(\ln 3))\,e^{-2} \approx .214\,K$ and

$\frac{\partial P}{\partial y} = K\,(3-2\sin x)\,(-2\cos y\sin y)\,e^{-z} = K\,(3-2\sin 2)\,(-2\cos(\ln 3)\sin(\ln 3))\,e^{-2} \approx -.495\,K$ and

$\frac{\partial P}{\partial z} = -K\,(3-2\sin x)\,(1+\cos^2 y)\,e^{-z} \approx -.737\,K$.

Hence $\frac{dP}{dt} = (.214\,K)\,(4) + (-.495\,K)\,(.5) + (-.737\,K)\,(1) = -.131\,K$ *pa/hour*.

LECTURE 18

another CHAIN RULE

S: I have to tell you that the title scares me!

P: But suppose we have $z = f(x,y)$ and each of x and y depend upon, say, two other variables u and v. Then we'd have $z = f(x,y)$ and $x = x(u,v)$, $y = y(u,v)$ and when either u or v change, so will x and y, hence so will z, so we might want to compute $\frac{\partial z}{\partial u}$ or maybe $\frac{\partial z}{\partial v}$.

S: When would that problem ever occur ... outside of this calculus course?

P: Well, maybe $z = f(x,y)$ and we want to switch to polar coordinates r and θ so $x = r \cos \theta$ and $y = r \sin \theta$ and that means x and y depend upon two other variables so that z ...

S: Yeah, I get it ... but that still sounds like a "math" problem, not a "real" problem.

P: Okay, suppose H(x,y) is the hardness (measured in some convenient unit!) at a point (x,y) in a flat sheet of material. Suppose, too, that a laser beam is cutting the material and the x- and y-position of the laser beam is controlled by two gears which rotate, the angles of rotation being given by θ_1 and θ_2. Then we'd have H(x,y) with $x = x(\theta_1,\theta_2)$ and $y = y(\theta_1,\theta_2)$. Now we want to compute the rate at which the hardness changes when, say, θ_2 changes. Got it?

S: Sounds like a pretty fishy "real" problem to me.

P: We're doing all this good stuff so if you ever run across a "real" problem you can at least say: "I remember doing problems like that." Of course they won't be exactly like the ones I invent, but is that really important? Techniques ... that's what we're learning. When you learn to play scales as a preamble to playing the piano, do you say: "Will these little songs ever occur outside of these piano lessons?"

S: Yes.

Consider the following problem:
$z = f(x,y)$ where $x = x(u,v)$ and $y = y(u,v)$ so that z is a function, indirectly, of both u and v. We want to compute $\frac{\partial z}{\partial v}$. That means we change <u>only</u> v (leaving u fixed) and deduce the change in z and take the limit $\lim_{\Delta v \to 0} \frac{\Delta z}{\Delta v}$ and, to do this, we let Δx and Δy be the changes in x and y when v changes by Δv and write (as we've done before!) the change in z as $\Delta z = \Delta z_x + \Delta z_y$ (these being the changes when x and y change one-at-a-time) so we get $\frac{\Delta z}{\Delta v} = \frac{\Delta z_x}{\Delta v} + \frac{\Delta z_y}{\Delta v} = \frac{\Delta z_x}{\Delta x}\frac{\Delta x}{\Delta v} + \frac{\Delta z_y}{\Delta y}\frac{\Delta y}{\Delta v}$ and now we can take the limits of all four parts and recognize the various partial derivatives (being careful to use "∂" rather than "d" when we're differentiating a function of more than a single variable) and we get: $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ and, in a similar manner, we can get: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$.

If $z = f(x,y)$ and $x = x(u,v)$, $y = y(u,v)$ then

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad \text{and} \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

I think that's enough on Chain Rules. I think you can see how it goes. If $K = f(x,y,z,u,v)$ and each of these five variables depend upon p, q, r and s, then $\frac{\partial K}{\partial r} = \frac{\partial K}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial K}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial K}{\partial z}\frac{\partial z}{\partial r} + \frac{\partial K}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial K}{\partial v}\frac{\partial v}{\partial r}$ and so on and so on and so on and so on ...

Example: Compute the indicated derivative(s):

(a) $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if $z = x^2y^3$ and $x = v \sin u$, $y = u^2e^v$

(b) $\dfrac{\partial P}{\partial t}$ if $P = \dfrac{C}{V}$ and $V = (x^2 + y^2) \sin \pi t$

Solution:

(a) $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} = (2xy^3)(v \cos u) + (3x^2y^2)(2ue^v)$ and

$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} = (2xy^3)(\sin u) + (3x^2y^2)(u^2e^v)$

(b) $\dfrac{\partial P}{\partial t} = \dfrac{dP}{dV}\dfrac{\partial V}{\partial t} = \left(-\dfrac{C}{V^2}\right)(\pi\,(x^2+y^2) \cos \pi t)$

In this last example, we write $\dfrac{dP}{dV}$ because P is a function of the single variable, V. There's no harm in writing $\dfrac{\partial P}{\partial V}$ of course. If someone asks "Why are you using the partial derivative?" you can always say "I'm keeping C constant." In fact, when dealing with a host of variables, it's not always clear from the notation $\dfrac{\partial P}{\partial V}$ just what is being held fixed. If P is pressure and V and T are volume and temperature, then it's often convenient to use $\left(\dfrac{\partial P}{\partial V}\right)_T$ to indicate a rate of change in pressure at constant temperature. You see this convention in thermodynamics which deals with gases ... among other things ... and every conceivable partial derivative $\left(\dfrac{\partial P}{\partial V}\right)_T$ or $\left(\dfrac{\partial V}{\partial T}\right)_P$ or $\left(\dfrac{\partial T}{\partial P}\right)_V$ seems to have its own name! Indeed, a book on thermodynamics is cover-to-cover partial derivatives.

Let's collect these Chain Rules:

a COLLECTION of CHAIN RULES

If $u = f(z)$ and $z = z(x,y)$ then $\dfrac{\partial u}{\partial x} = \dfrac{du}{dz}\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial u}{\partial y} = \dfrac{du}{dz}\dfrac{\partial z}{\partial y}$

If $P = f(x,y,z)$ and $x = x(t)$, $y = y(t)$ and $z = z(t)$, then $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial P}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial P}{\partial z}\dfrac{dz}{dt}$

If $z = f(x,y)$ and $x = x(u,v)$, $y = y(u,v)$ then $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$ and $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

P: When you see $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial P}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial P}{\partial z}\dfrac{dz}{dt}$ does it remind you of anything?

S: Uh ... it's something like ... uh ... no, it doesn't.

P: It has the form AB + CD + EF, doesn't it?

S: Yes! The DOT product, right? In fact, AB + CD + EF = [A, C, E] • [B, D, F], the DOT product between these two vectors. Am I a genius or what?

P: Very good. Now write $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial P}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial P}{\partial z}\dfrac{dz}{dt}$ as a dot product.

S: Easy. It's $\left[\dfrac{\partial P}{\partial x}, \dfrac{\partial P}{\partial y}, \dfrac{\partial P}{\partial z}\right] \bullet \left[\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right]$.

P: These two vectors, $\left[\dfrac{\partial P}{\partial x}, \dfrac{\partial P}{\partial y}, \dfrac{\partial P}{\partial z}\right]$ and $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right]$, do they remind you of anything? Do you recognize them? Think.

S: One's the gradient ... but in 3-D, and the other is ... is ...

P: The velocity of a moving point (x,y,z), assuming t is the time.

S: So I can write $\frac{dP}{dt}$ = *grad* P • **V** where V is the velocity. But what does it mean?

P: It depends upon the problem. If the position of a plane is changing with time, say x = x(t), y = y(t), z = z(t), and P(x,y,z) is the air pressure at the position (x,y,z), then the plane will experience a changing pressure and it will change at the rate *grad* P • **V** where *grad* P is the local gradient (at the position of the plane) and **V** is the plane's velocity vector at the time t. Nice, eh? In fact ...

S: In fact I can tell you how the plane should move so as to experience the greatest increase in pressure! We'd want **V** in the direction of *grad* P. Not only that, but the length of *grad* P would actually give this maximum rate of change of pressure. In fact I can tell you right now that the maximum value of $\frac{dP}{dt}$ is $\sqrt{\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial P}{\partial x}\right)^2}$. What a genius!

P: Well ... not quite: $\frac{dP}{dt}$ is measured in *pascals/hour* whereas *grad* P is measured in *pascals/kilometre* and so is $\sqrt{\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial P}{\partial x}\right)^2}$ so you need to multiply the length of *grad* P by something with the dimensions of ... what?

S: That's confusing.

P: No it's not. To get *pascals/hour* you multiply *pascals/kilometre* by *kilometres/hour*. Where do you get this *kilometres/hour*?

S: Uh ... *kilometres/hour* ... that's **V**, right?

P: Right. Notice that $\frac{dP}{dt}$ = *grad* P • **V** = | *grad* P | • | **V** | cos θ where θ is the angle between these two vectors and to get the maximum $\frac{dP}{dt}$ (measured in *pascals/hour*) you fly your plane so θ = 0 and that gives a maximum ...

S: The maximum is | *grad* P | • | **V** | ... the length of *grad* P times the length of **V**. Nice!

P: Now let me show you something interesting ...

Directional Derivatives, revisited

When we considered the rate of change of f(x,y) in some α-direction (making an angle α with the positive x-axis), we measured distance in this α-direction with the variable "s" and obtained, eventually: $\boxed{\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha}$. Now let's do the following. We let z = f(x,y) and let x and y move from (a,b) in the α-direction along the line: x = a + s cos α, y = b + s sin α. (Parametric equations for the line through (a,b) with direction α). To compute the directional derivative we want $\lim_{\Delta s \varnothing 0}\frac{\Delta z}{\Delta s}$ but now we have a Chain Rule to cover this situation: z = f(x,y) and x = a + s cos α, y = b + s sin α gives $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{dx}{ds}+\frac{\partial z}{\partial y}\frac{dy}{ds}=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\sin\alpha$ which is precisely the result we obtained before ... which gives us a warm feeling.

S: Hey! Why do you call it $\frac{\partial z}{\partial s}$?

P: That's what we've been calling it.

S: But you should write it as $\frac{dz}{ds}$, right? I mean, we're talking about a function of the single variable "s", right?

P: Uh ... you have a point there. Yes, I guess we should call it $\frac{dz}{ds}$. Hmmm. Why didn't I think of that? Well, I suppose it made sense when we started all this since we were buried in variables and ...

S: Excuses, excuses.

P: Okay, from now on I'll call it $\frac{df}{ds}$. Happy? Now watch this and I'll show you something really wonderful:

Implicit Differentiation, revisited

Way back when, we wanted to find $\frac{dy}{dx}$ given some relation f(x,y) = C which didn't define y explicitly as a function of x ... so we invented "implicit differentiation". Watch carefully as we go through the steps:

Example: Determine $\frac{dy}{dx}$ if $x^2\sin y + e^x \ln(xy) = 1$

Solution: Differentiating $\frac{d}{dx}\left\{ x^2\sin y + e^x(\ln x + \ln y) = 1 \right\}$ gives

$$x^2\cos y\frac{dy}{dx} + 2x\sin y + e^x \ln(xy) + e^x\left(\frac{1}{x}\right) + e^x\left(\frac{1}{y}\frac{dy}{dx}\right) = 0.$$

Now we collect terms:

$$\left\{x^2\cos y + \frac{e^x}{y}\right\}\frac{dy}{dx} + \left\{2x\sin y + e^x\ln(xy) + \frac{e^x}{x}\right\} = 0.$$

Now we solve for $\frac{dy}{dx} = -\dfrac{2x\sin y + e^x\ln(xy) + \frac{e^x}{x}}{x^2\cos y + \frac{e^x}{y}}$.

P: Do you recognize any of the terms, like maybe the numerator or denominator?
S: Wait ... uh, no.
P: The numerator is just $\frac{\partial z}{\partial x}$ and the denominator is $\frac{\partial z}{\partial y}$.
S: Huh? Who's z?
P: Oh, sorry, $z = x^2\sin y + e^x\ln(xy)$ and we're on some level curve z = 1, and if we want the slope of the tangent line to this curve we can find $\frac{dy}{dx}$ by implicit differentiation OR (and this is the nice part) we can just use: $\frac{dy}{dx} = -\dfrac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$.

$$\boxed{\text{To determine } \frac{dy}{dx} \text{ implicitly from } f(x,y) = C, \text{ use } \frac{dy}{dx} = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}}$$

S: So ... one example proves it, right?
P: Watch this. We have z = f(x,y) = C and we're going to stay on this level curve, so if we change x then y must change in some particular way in order to keep f(x,y) equal to C so we don't move off the curve. In other words, y must *follow* the level curve, so y is a function of x and of course x is a function of x (what else?). Okay, we write this out like so: z = f(x,y) and x = x, y = y(x). See? We've said that y is a function of x by writing y = y(x) and just by looking at these equations you can see that we've got a Chain Rule going here. We have the function f(x,y) where each of x and y is a function of x. Hence: $\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$ because z = 1 so clearly $\frac{dz}{dx} = 0$. (If we don't have this we don't stay on the level curve so when we get some $\frac{dy}{dx}$ it won't be the slope we're looking for!) Now we put $\frac{dx}{dx} = 1$ and solve for $\frac{dy}{dx} = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ and voila! A simple expression for implicit differentiation. Now go to all your friends and defy them to beat you in the game of Implicit Differentiation.

the GRADIENT vector is normal to the level curve

We've been saying for some time now that:

(1) if we stand at the point P(a,b) on a level curve f(x,y) = C, namely the level curve f(x,y) = f(a,b), and

(2) we evaluate *grad* f at P(a,b) on this level curve, namely $\left[\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right]$, then

(3) this gradient vector is normal to the curve.

We're now in a position to do something more than just illustrate this with examples and pictures.

On the level curve f(x,y) = C the slope of the tangent line at P(a,b) is $\boxed{\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}}$ where each of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is to be evaluated at (a,b). On the other hand the gradient vector is $\left[\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right]$ and has a direction characterized by the slope: $\boxed{\frac{\text{y-component}}{\text{x-component}} = \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}}$. Just stare at those two slopes. They're negative reciprocals! That means *grad* f is perpendicular to the tangent line. In other words, *grad* f is NORMAL to the level curve!

There's another way to look at this. We can write ...

S: Another way? Why another way?
P: I really don't like equations like the above which are good in 2-dimensions but it's hard to see what the analogous equations would be in 3-dimensions, or 4 or 5.
S: Okay, forge ahead!

One awkward thing about the "tangent line": it isn't clear what we should do in 3-D since there are infinitely many tangent lines to a surface at a point P(a,b,c) on the surface so we won't be able to say "the gradient has a slope perpendicular to the slope of the tangent line". On the other hand there is only ONE normal direction to the surface (well ... two actually, pointing in opposite directions!). It would be nice if we could show that, for the *3-dimensional analogue*, we'd still have *grad* f pointing in a normal direction. Of course, for the *3-dimensional analogue* we must consider "level curves" f(x,y,z) = C rather than f(x,y) = C and these aren't even curves but surfaces ... but that's only a change in wording, so we can call f(x,y,z) = C "level surfaces" if we wish.

Now, is we stand at a point P(a,b,c) and compute the vector *grad* f = $\left[\frac{\partial f}{\partial x}(a,b,c), \frac{\partial f}{\partial y}(a,b,c), \frac{\partial f}{\partial z}(a,b,c)\right]$, will it be perpendicular to the level surface which passes through P, namely f(x,y,z) = f(a,b,c)? The answer is YES! In fact, it will be perpendicular to every tangent line to the surface at the point P(a,b,c)! In fact, it will be perpendicular to ... to what?

S: Huh? I don't understand the question.
P: In order to be perpendicular to every possible tangent line the gradient vector must be perpendicular to what? Tell me what contains every single tangent line.
S: I still don't understand ...
P: The TANGENT PLANE at P(a,b,c)!!

LECTURE 19

the TANGENT PLANE

We must first investigate the nature of planes else how would we recognize one when we see its equation? To see what form the equation of a plane must have, we go over lines in 1-dimensional calculus ... carefully, trying to put things in a form which generalizes to 3-dimensions.

S: Why lines? Why not parabolas or circles or ...
P: Because the 3-dimensional analogue of a line is a plane! Pay attention. It'll be clear soon enough.
S: Come now. Lines are lines and the 3-D line is just as much a line as a 2-D line so why isn't the 3-D line just a line. I mean,

why is the 3-D analogue a plane? It doesn't make sense. I mean ...

P: Pay attention!!

Recall that the equation of a line in 1-variable calculus has the form $Ax + By = C$. If the line is to pass through a given point $P(x_0, y_0)$, then $Ax_0 + By_0 = C$ so this gives the required value of C, and <u>all</u> lines through (x_0, y_0) have the form $Ax + By = Ax_0 + By_0$, or, to put it more elegantly:

$$\boxed{A(x - x_0)+B(y - y_0) = 0}\ .$$

We immediately (!) recognize this as having the form of a DOT product: the DOT product between the vector $[A, B]$ and the vector $[x - x_0, y - y_0]$.

Who are these vectors?

Since $P(x_0, y_0)$ is a fixed point on our line and (x,y) is a variable point, then the vector $[x - x_0, y - y_0]$ just points along the line, from $P(x_0, y_0)$ to (x,y). It's a "tangent vector". In fact, if we take $\dfrac{\text{y-component}}{\text{x-component}}$ we get $\dfrac{y - y_0}{x - x_0}$ which, of course, is the slope of the line (and the slope of the tangent vector!).

Now, how about $[A, B]$? Since the DOT product $A(x - x_0)+B(y - y_0) = 0$, then $[A,B]$ must be perpendicular to the tangent vector $[x - x_0, y - y_0]$, hence $[A, B]$ must be a NORMAL vector!

In fact, if we write the equation of a line in the form $y = mx + k$, then put it into the form $Ax + By = C$ (i.e $m x - y = - k$), the vector $[A, B]$ is $[m, -1]$ and this is a normal vector. In fact, the slope of this normal is $\dfrac{\text{y-component}}{\text{x-component}} = -\dfrac{1}{m}$ which is certainly the negative reciprocal of the slope of the line.

Also, the equation of the tangent line to a curve $y = f(x)$ is $y = f(a) + f'(a)(x - a)$ and, in our "standard" form $Ax + By = C$, this reads $f'(a)\,x - y = \text{constant}$ so a NORMAL vector is $[f'(a), -1]$ which, again, has slope $-\dfrac{1}{f'(a)}$, perpendicular to the line.

In general if we wish to construct a line we can begin with a point (x_0, y_0) and a NORMAL direction $[A, B]$ and ask "Where are all the points (x,y) which make $[x - x_0, y - y_0]$ perpendicular to $[A, B]$?" The answer? They must satisfy $A(x - x_0)+B(y - y_0) = 0$, and <u>that's</u> the equation of a line.

Now on to planes in 3-D ... and we'll see why *planes* are the natural 3-D analogue of the line:

We consider a point (x_0, y_0, z_0) and a NORMAL direction, say $[A, B, C]$ (which, of course, is now a vector in 3-D with 3 components). "Where are all the points (x,y,z) which make $[x - x_0, y - y_0, z - z_0]$ perpendicular to $[A, B, C]$?" The answer? They must satisfy $A(x - x_0)+B(y - y_0) + C(z - z_0) = 0$, and <u>that's</u> the equation of a plane.

> The equation of a plane through (x_0, y_0, z_0) with normal vector $[A, B, C]$ is
> $$A(x - x_0)+B(y - y_0) + C(z - z_0) = 0$$

Okay, now that we recognize the equation of a plane when we see it, let's find the TANGENT plane to some surface $z = f(x,y,)$ at the point $P(a,b,c)$ where, of course, $c = f(a,b)$ (else our point wouldn't lie on our surface!). First of all, it must have the form $A(x - a)+B(y - b) + C(z - c) = 0$. This represents every possible plane through (a,b,c).

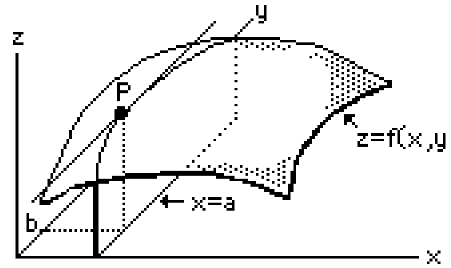
Who are A, B and C?

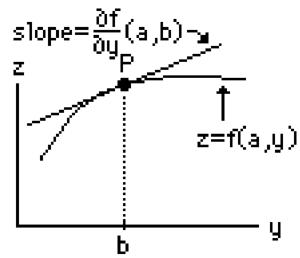
S: The normal to the plane!

P: Yes, yes,of course, but if we have the surface $z = f(x,y)$ and the point $P(a,b,c)$ how do we calculate A, B and C?

How to find A, B and C so the plane through (a,b,c) is "tangent" to the surface $z = f(x,y)$?

One thing we've known for some time is the slope of two particular tangent lines to the surface $z = f(x,y)$.





If we look at the intersection of the surface z = f(x,y) with the plane x = a (from a vantage point along the positive x-axis, looking back, so the z-axis goes up and the y-axis goes to the right) we see a curve z = f(a,y) and the tangent line to this curve must lie in our plane A (x - a)+B (y - b) + C (z - c) = 0. In fact, since we're in the plane x = a, every point on this tangent plane must satisfy B (y - b) + C (z - c) = 0 which is where the tangent plane intersects the plane x = a ... and this intersection <u>is</u> the tangent line to the curve z = f(a,y) at the point P. But we already know how to calculate the equation of the tangent line to z = f(a,y) when y = b. It's $z = f(a,b) + \frac{\partial f}{\partial y}(a,b)\,(y - b)$, or to put it into a more elegant form, it's $\frac{\partial f}{\partial y}(a,b)\,(y - b) + (-1)\,(z - c) = 0$ where we've put f(a,b) = c. Comparing with B (y - b) + C (z - c) = 0 we see that B and C must be proportional to $\frac{\partial f}{\partial y}(a,b)$ and (-1) respectively. Or, to put it differently, if we find $\frac{\partial z}{\partial y}$ from B (y - b) + C (z - c) = 0 by implicit differentiation ... giving $B + C\frac{\partial z}{\partial y} = 0$ ... it must be the same as $\frac{\partial f}{\partial y}(a,b)$ and that means that $\boxed{-\frac{B}{C} = \frac{\partial f}{\partial y}(a,b)}$.

S: Whoa! Proportional you say? Why aren't they equal? I mean, why isn't $B = \frac{\partial f}{\partial y}(a,b)$ and C = -1?

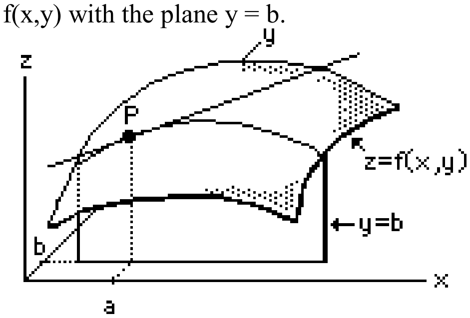
P: I give you the equations 2 x + 4 y = 8 and x + 2 y = 4. They're the same line, right? (I'm talking about lines in the x-y plane.)

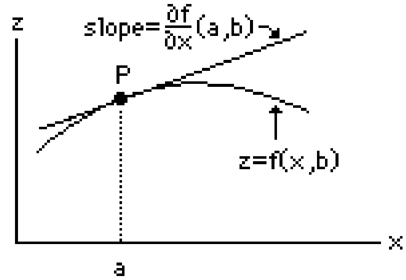
S: Yeah, they're the same line.

P: So are the coefficients exactly the same? Is 2 = 1 and 4 = 2 and 8 = 4? No. They're just proportional. After all, you can multiply the equation of a line by some constant k and get the same line and all the coefficients will get multiplied by k so the new coefficients won't be the same but they <u>will</u> be proportional with constant of proportionality equal to k.

S: Okay, okay, okay!

Now, having obtained $B = k\frac{\partial f}{\partial y}(a,b)$ and C = - k we go to our <u>other</u> tangent line ... the one which we get by intersecting z = f(x,y) with the plane y = b.





We repeat the procedure: the tangent plane A (x - a)+B (y - b) + C (z - c) = 0 intersects the plane y = b in the line A (x - a) + C (z - c) = 0 which must therefore be the tangent line to the curve z = f(x,b) at P, namely $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,(x - a)$ or, more elegantly, $\frac{\partial f}{\partial x}(x - a) + (-1)\,(z - c) = 0$ where again we've put f(a,b) = c, and again we compare the two equations and see that A is proportional to $\frac{\partial f}{\partial x}(a,b)$ and C proportional to -1. Or, to put it differently, if we find $\frac{\partial z}{\partial x}$ from A (x - a) + C (z - c) = 0 by implicit differentiation ... giving $A + C\frac{\partial z}{\partial y} = 0$ ... it must be the same as $\frac{\partial f}{\partial x}(a,b)$ and that means that $\boxed{-\frac{A}{C} = \frac{\partial f}{\partial x}(a,b)}$.

We collect all our results:

$-\frac{B}{C} = \frac{\partial f}{\partial y}(a,b)$ and $-\frac{A}{C} = \frac{\partial f}{\partial x}(a,b)$ and that means that $A = -C\frac{\partial f}{\partial x}(a,b)$ and $B = -C\frac{\partial f}{\partial y}(a,b)$ so the tangent plane $A(x-a)+B(y-b)+C(z-c)=0$ becomes $-C\frac{\partial f}{\partial x}(a,b)(x-a) - C\frac{\partial f}{\partial y}(a,b)(y-b) - C(z-c) = 0$.

We then have:

> The TANGENT PLANE to $z = f(x,y)$ at the point $P(a,b,c)$ is:
>
> $$\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) - (z-c) = 0$$
>
> $$\text{or } z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

See how the equation of the tangent *plane* mimics the equation of the tangent *line* to $y = f(x)$, in 2-D?

$\boxed{y = f(a) + f'(a)(x-a)}$ versus $\boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)}$

It seems clear that a "surface" $u = f(x,y,z)$ in 4-dimensions has a tangent "plane" at $x=a$, $y=b$, $z=c$ given by $u = f(a,b,c) + \frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c)$ and so on and so on and so on ...

S: Question: there seems to be a lack of symmetry here. I mean, why does z get the (-1) and x and y get the partial derivatives. I mean ...

P: Very nice observation! In fact we've singled out z for special consideration simply by writing $z = f(x,y)$. That destroys the symmetry right there! It's like finding the tangent line to $y = f(x)$ and writing it as $y = f(a) + f'(a)(x-a)$ or "more elegantly" (trying to regain the loss of symmetry): $f'(a)(x-a) + (-1)(y - f(a)) = 0$. See? y gets the (-1) and x gets the derivative. On the other hand, if you had started NOT with $y = f(x)$ ... which singles out y for special consideration ... but collected the x's and y's together, writing $f(x,y) = C$, we'd get a more symmetrical equation for the tangent line. For example we start with $x^2 + y^2 = C$ and ask for the tangent line at some point on this circle and we don't "solve for y" but leave it in this symmetrical form. Do you know the equation of the tangent line at some point (a,b) on $x^2 + y^2 = C$?

S: Uh ... it shouldn't be that hard. Let's see ... I have the point (a,b) so I need the slope and I get that by implicit differentiation, but now I'll impress you by using $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ where $f = x^2 + y^2$ and I get $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{a}{b}$ at the point in question so the tangent line is $\frac{y-b}{x-a} = -\frac{a}{b}$ which is the same as $ax + by = a^2 + b^2$. Hey! That's pretty nice!

P: And, of course, $a^2 + b^2 = C$ because (a,b) is on the curve $x^2 + y^2 = C$. But check it out ... your equation, I mean.

S: Huh?

P: Does (a,b) lie on your line? Are the dimensions correct? At the point (a,0) do you get a vertical line? At (0,b) ...

S: Yeah, okay ... let's see .. if everything is in *metres* then "ax" and "by" and "a^2+b^2" are all in *metres*^{2} so that checks ... and, uh ... everything else checks out too. So what's the equation of a tangent plane in a symmetrical form?

P: Figure it out yourself.

S: Okay, pay attention:

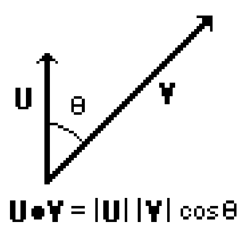
Rather than writing $z = f(x,y)$ we write $F(x,y,z) = C$. That's a surface, but all variables are collected on the left of the equation and everything is nice and symmetrical. We're at a point P(a,b,c). We want the tangent plane at P. We have the point and we now need ... uh ... a direction, the NORMAL direction, the vector [A, B, C], then we plug into $A(x-a) + B(y-b) + C(z-c) = 0$. Okay, the normal to $F(x,y,z) = 0$ is ... is ... well, I'll generalize from the 2-D problem. For the level curve f(x,y) = C, the normal vector is $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$ so for the 3-D level surface F(x,y,z) = 0 the normal will be $[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}]$ so our tangent plane is $\frac{\partial F}{\partial x}(x - a) + \frac{\partial F}{\partial y}(y - b) + \frac{\partial F}{\partial z}(z - c) = 0$.

P: Excellent! Except that the partial derivatives, $\frac{\partial F}{\partial x}$ etc., must be evaluated at the point P(a,b,c). They're numbers, after all. The only variables in your equation should be the x, y and z in (x-a), (y-b) and (z-c). And notice what happens if you start with z = f(x,y) and put everything on the one side of the equation? You'd get F(x,y,z) = f(x,y) - z = 0 so the normal would be $[\frac{\partial}{\partial x}F(x,y), \frac{\partial}{\partial y}F(x,y), \frac{\partial}{\partial z}F(x,y)]$ and that's just $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1]$ so you can see where the (-1) comes from!

The tangent plane to the surface F(x,y,z) = C at the point P(a,b,c) is

$$\frac{\partial F}{\partial x}(a,b,c)\,(x - a) + \frac{\partial F}{\partial y}(a,b,c)\,(y - b) + \frac{\partial F}{\partial z}(a,b,c)\,(z - c) = 0$$

S: I've noticed that you don't have too many pictures. What ever happened to your picture-worth-a-thousand-words?



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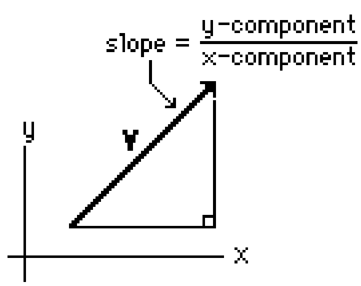

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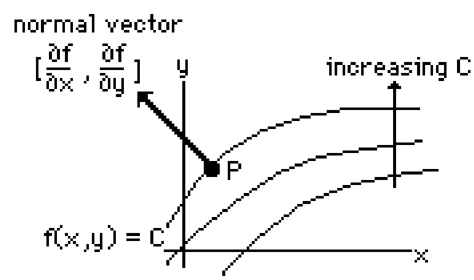
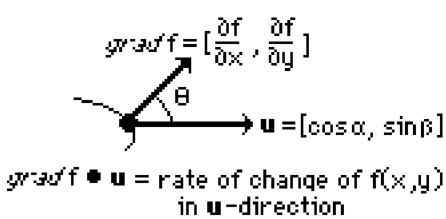
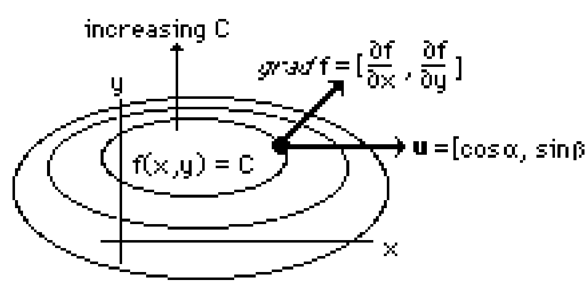


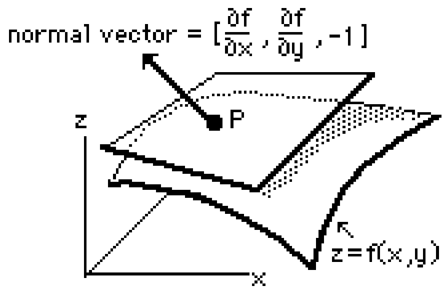
the normal to a level curve

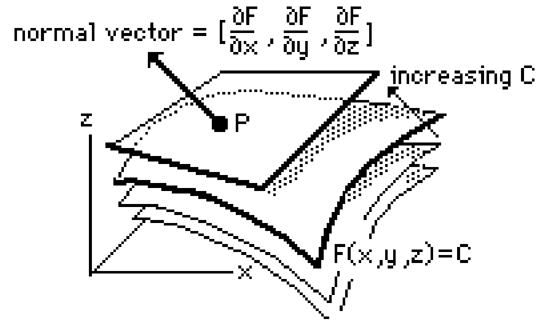


the directional derivative



grad f is normal to the level curve f(x,y) = C





the normal to a level surface

grad F is normal to the level surface F(x,y,z) = C

P: And I've noticed you haven't asked if this will be on the final exam.

LECTURE 20

OPTIMIZATION

S: I kind of like that symmetry stuff, like ax + by = C is the tangent line to $x^2 + y^2 = C$ at P(a,b). What else can you do with symmetry?

P: Well ... let's see you solve this problem. You are to construct a rectangular wall of 100 *metres*^{2} and around this wall you'll put a gold border. Gold is expensive, so you want the length of the border to be a minimum. What should be the dimensions of the wall?

S: Easy! I let the width and height of the wall be x and y so I want L = 2x + 2y to be a minimum; that's the border. There are two variables so I look for a relation between them, and it's xy = 100 because that's the required area. That means $y = \frac{100}{x}$ so the length of the border is $L = 2x + \frac{200}{x}$ and I minimize this function of a single variable, x. First I find $\frac{dL}{dx} = 2 - \frac{200}{x^2}$ and see that it's negative when x is small because $-\frac{200}{x^2}$ is large and negative. In fact, for small x, $L \approx \frac{200}{x}$ and that's a decreasing function. Then I see that $\frac{dL}{dx}$ is positive for x large because $-\frac{200}{x^2}$ is smaller than 2. In fact, for large x, $L \approx 2x$ and that's an increasing function. That means the graph goes down, then up, so when $\frac{dL}{dx} = 2 - \frac{200}{x^2} = 0$ I have my minimum and that means that $x^2 = 100$ so x = 10. Since xy = 100, then y = 10 too. It's a square wall. The width is x = 10 and the height is y = 10.

P: Okay, now suppose you call the <u>width</u> y and the <u>height</u> x. That's the opposiite to the last time. Do the problem again.

S: You're trying to tell me something, right? Okay, I have L = 2x + 2y just like before and xy = 100 just like before and ... isn't everything going to be the same?

P: Precisely! You'll get y = 10 and x = 10 and how could you get anything else? The problem is perfectly symmetrical in x and y ... so the answer must be symmetrical too ... so x and y have to be the same. You could have predicted this without any calculus at all. In fact, if you lie on your side and look at the wall, the height looks like a width and the width looks like a height. In fact, if you were just talking about a rectangle, who's to say what's the height and what's the width? Now suppose you're building a closed box of side lengths x, y and z and suppose the volume had to be 100 *metres*^{3} and the amount of cardboard had to be minimized. Then you'd want the area of all six sides, namely A = 2xy + 2yz + 2zx, to be minimum and you'd want the volume to be xyz = 100. Now find the dimensions which minimize A.

S: They're all the same, right?

P: Sure! And I didn't even tell you which was the height or width or length. It makes no difference. These labels are <u>ours</u>. The math doesn't know height from width. If the math says "x = 5, y = 5 and z = 4" and you build the box then <u>you</u> have to decide what's the height and width and length. When you're finished I can turn the box on its side and now I've got a *different* height, width and length. Hence, unless there are several solutions to the problem, the answer *must* be symmetrical in x, y and z and that means a cubical box ... and that means "x = 5, y = 5 and z = 4" is wrong, you made a mistake ... and you don't need calculus to see that. Can you imagine solving this problem for a box manufacturer and you tell the boss "x = 5, y = 5 and z = 4" and he says you're all wet because it's symmetrical so x = y = z and he didn't even graduate from high school! See? Don't rely too heavily on the math: it isn't as smart as people.

S: Can I solve that problem? I mean the box problem. We've got too many variables, I mean ...

P: I'm glad you asked that question ...

Whereas we've reproduced many of the ideas of 1-variable calculus, generalizing to higher dimensions, one we haven't tackled is the optimization or max/min problem. We'll do the "box problem" to see what difficulties arise:

Example: A 100 m^3 box is to be made with minimum cardboard. What are its dimensions?

Solution: If the side lengths are x, y and z then the amount of cardboard is $A = 2xy + 2yz + 2zx$, but these variables are related by $xyz = 100$. If we put $z = \frac{100}{xy}$ then we still have $A = 2xy + 2y\left(\frac{100}{xy}\right) + 2\left(\frac{100}{xy}\right)x$ which is a function of two variables. *Any* (positive) values for x and y will give a box, and the volume will definitely be 100 *metres*^{3} if we choose $z = \frac{100}{xy}$, so x and y are definitely "independent" variables ... so what's the best choice for x and y ... so A is a minimum?

We stare at $A = 2xy + \frac{200}{x} + \frac{200}{y}$ and realize that it's a surface and what we want is the lowest point on this surface ... where we must ignore any parts of the surface where $x < 0$ or $y < 0$.

S: It's symmetrical in x and y so x = y!

P: Yes, but we want to generate a methodology for solving this type of problem and they won't *all* be symmetrical. In fact, if one side of our box was open, then that'd destroy the symmetry.

What characterizes the minimum point on this surface, $A = 2xy + \frac{200}{x} + \frac{200}{y}$? Perhaps we should first agree on what we *mean* by a "minimum" and maybe that'll help. By a minimum we mean a point P(a,b) where, if x and/or y change, then A(x,y) gets larger ... or at least doesn't get smaller. That means the surface is flat at a minimum point. We're in a valley ... mountains all around ... every direction is up. How can we say that mathematically?

S: The tangent line is horizontal ... <u>every</u> tangent line is horizontal ... I mean the tangent PLANE is horizontal!

P: And how do we say that ... mathematically?

S: The normal points straight up!

P: And how do we say that ... mathematically?

S: Uh ... why not compute the normal vector and see where it points straight up ... at what (x,y) points?

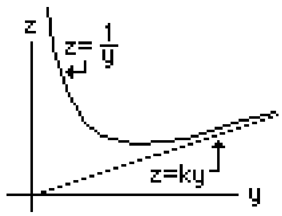
P: Not <u>the</u> normal vector ... but <u>a</u> normal vector. Remember, there are an infinite number of them, although they all point in the same direction ... or the opposite direction. When we compute a normal vector using partial derivatives, we're just getting one normal. What we want is for it to point up ... or down ... it doesn't matter which. Wherever this happens we have a candidate for a maximum or a minimum.

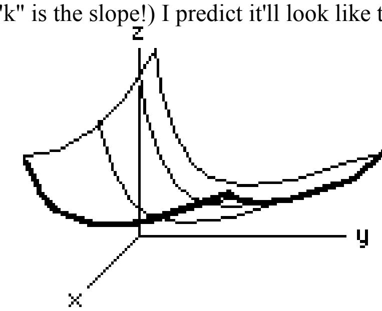
For $A = 2xy + \frac{200}{x} + \frac{200}{y}$ a normal vector is $\left[\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, -1\right] = [2y - \frac{200}{x^2}, 2x - \frac{200}{y^2}, -1]$ and it must have zero x- and y-components in order to point up or down. (In fact we have little choice; the z-component is -1 so this normal points down whether we like it or not!) Hence we want $\boxed{2y - \frac{200}{x^2} = 0}$ and $\boxed{2x - \frac{200}{y^2} = 0}$ and that gives us two equations in two unknowns to solve for x and y (and if there's more than one solution we're in trouble!).

We have $y = \frac{100}{x^2}$ from the first and $x = \frac{100}{y^2}$ from the second and substituting "y" into the second gives $x = \frac{x^4}{100}$ or $x(x^3 - 100) = 0$ so either $x = 0$ or $x = 100^{1/3}$. Clearly $x = 0$ will not provide our minimum area (nor can it provide a volume of 100 m^3!). The solution is $x = 100^{1/3}$. Since $y = \frac{100}{x^2}$, that gives $y = 100^{1/3}$ as well. Finally, since $z = \frac{100}{xy}$, that gives $z = y = 100^{1/3}$ as well. (Now we check to see that xyz = 100 ... which it is.)

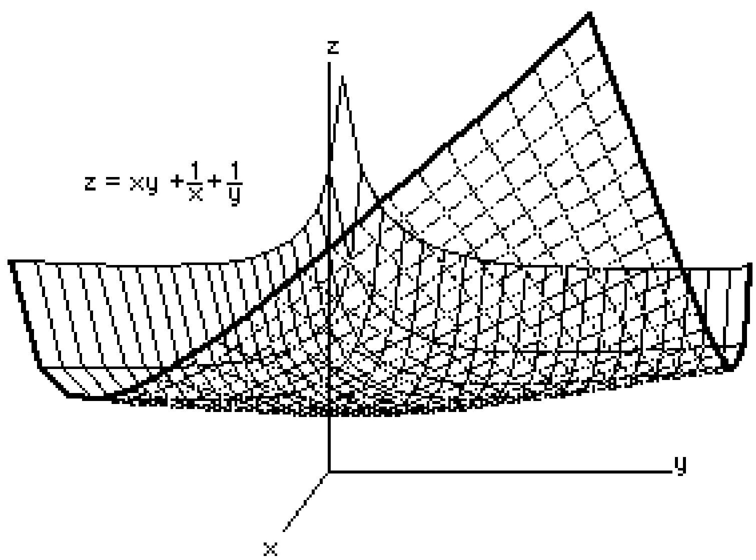
S: Why does the math give that x = 0 solution? I know that it doesn't make sense for our box because that'd make $y = \frac{100}{x^2}$ infinite. But the surface must look pretty weird, right?

P: Let's plot it. In fact, I won't plot $A = 2xy + \frac{200}{x} + \frac{200}{y}$, I'll plot $z = xy + \frac{1}{x} + \frac{1}{y}$. It'll have the same features. In fact, we'll first try to predict what it'll look like. When I intersect this surface by planes x = k, I get a curve $z = ky + \frac{1}{k} + \frac{1}{y}$ which, in the y-z plane looks like the hyperbola $z = \frac{1}{y}$ when y is small, and like the line z = ky when y is large. As we move along the x-axis, slicing the surface by planes x = k with ever increasing k, the curve of intersection continues to look like $z = \frac{1}{y}$ when y is small, but the line z = ky has larger and larger slope. ("k" is the slope!) I predict it'll look like the graph below.





S: Okay, let's see it.



P: Note that the edges of the surface, in the diagram, are intersections with planes x = A or y = B (for certain constants A and B). The former is $z = Ay + \frac{1}{A} + \frac{1}{y}$ and the latter is $z = Bx + \frac{1}{x} + \frac{1}{B}$. Each has that characteristic flavour I mentioned ... a hyperbola when x or y is small and a straight line when they are large. Nice, eh?

S: So why does the math give x = 0 as a place where the normal is straight up? That was my original question, remember?

P: It doesn't. For $A = 2xy + \frac{200}{x} + \frac{200}{y}$ the normal vector is $[2y - \frac{200}{x^2}, 2x - \frac{200}{y^2}, -1]$ and when either x or y is small it has a very large and negative x- and y-component so it's almost parallel to the x-y plane. In other words, since the surface rises very steeply, the normal vector is almost horizontal.

S: But when you put these x- and y-components equal to zero ... so the normal would go straight up ... or straight down ... and you solved the equations, you got x = 0!

P: And y = ∞! That's NOT a solution!

S: But why did the math do that to us ... uh, to <u>you</u>?

P: Okay. We asked "At what points (x,y) are $2y - \frac{200}{x^2} = 0$ <u>and</u> $2x - \frac{200}{y^2}$?" In trying to solve these two equations we got $x = \frac{x^4}{100}$ which seems to imply a solution x = 0 ... but that's NOT a solution to these two equations. I know, I know ... why did the math do it to us? If you want to see something even scarier, we could write these two equations in the form $x^2y = 100$ and $xy^2 = 100$ so that means that $x^2y = xy^2$ which can be written xy (x - y) = 0 so x = 0 OR y = 0 OR x = y. It's only the last that gives a solution to the original equations! You have to be careful when you manipulate equations. You could introduce "solutions" which don't satisfy the original equations. If we think that xy (x - y) = 0 has the same solutions as the pair of equations: $2y - \frac{200}{x^2} = 0$, $2x - \frac{200}{y^2}$ then we're mistaken. Of course some students would get <u>more</u> solutions, like $x = 0$, $y = \infty$ and they'd justify these solutions by substituting into $2y - \frac{200}{x^2} = 0$ and get $2(\infty) - \frac{200}{0^2} = \infty - \infty = 0$ so it satisfies *that* one, then they'd substitute into $2x - \frac{200}{y^2}$ and get $2(0) - \frac{200}{\infty^2} = 0 - 0$ (because everybody knows that $\frac{200}{\infty^2} = 0$) and it satisfies *that* one too. That's why I like to "check it out" when I get a solution. It has to be reasonable. Maybe I can check the dimensions. Maybe I can estimate the answer. Maybe I can solve the problem in a different manner. Maybe ...

S: Okay ... gotcha.

P: Before we forget, let's indicate how we solved this optimization problem:

To maximize or minimize z = f(x,y), we looked for points where the NORMAL vectors are parallel to the z-axis (pointing straight up or down). Since the normal vector is $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1]$, we look at points (x,y) where the x- and y-components are zero (and that will give only a z-component and it will then point up or down). That is, we solve the two equations in two unknowns: $\boxed{\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0}$. We make a note of this:

> To maximize or minimize f(x,y), look for points where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

S: Just "look at points"? That's all?

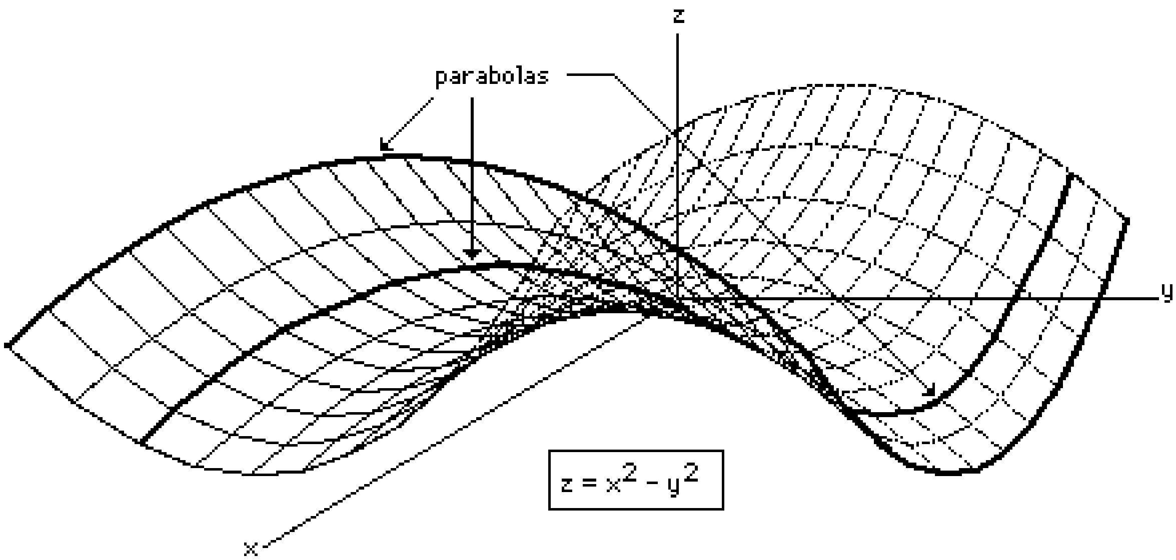
P: If we solve these two equations and find a solution (x_0, y_0) we can't guarantee that we have a minimum or maximum.

S: But if the normal points up ...

P: Let me give you an example. Remember the hyperbolic paraboloid? Its equation is $z = f(x,y) = x^2 - y^2$. Now we solve $\frac{\partial f}{\partial x} = 2x = 0$ and $\frac{\partial f}{\partial y} = -2y = 0$ and we get the point (0,0). The normal does indeed point up ... actually it points down ... I have to remember that ... it's $[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1]$ and the z-component is negative meaning it points down. Anyway, if you stand at the origin and move in the x-direction you go UP the surface but if you move in the y-direction you go DOWN the surface. It's flat at the origin but you don't have either a maximum or a minimum.

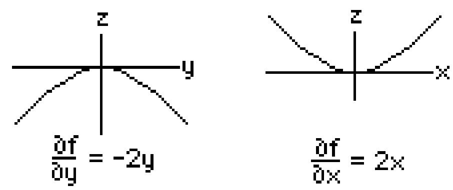
S: How about a picture?

P: Okay, but look at the origin. It's a so-called SADDLE POINT. Go in one direction and the surface goes down. Go in an orthogonal direction and it goes up. It's neither a maximum nor a minimum.



S: Is there some kind of test? A first derivative test or something?

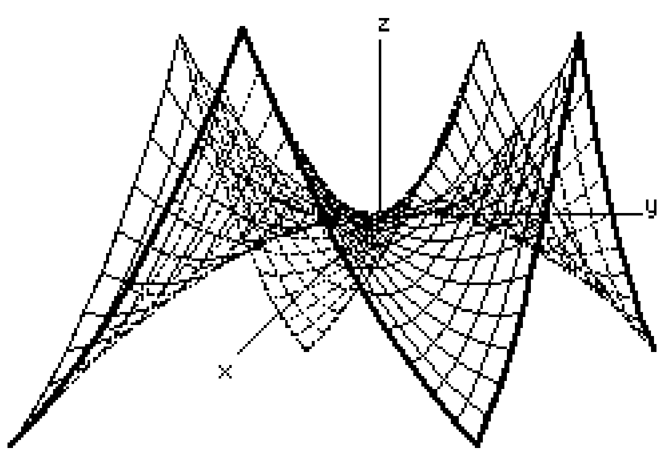
P: We can, of course, look at the partial derivative with respect to x, that's $\frac{\partial f}{\partial x} = 2x$, and notice that it's negative for $x < 0$ and positive for $x > 0$ but that only tells us that the surface comes down then goes up as x increases. The other partial, $\frac{\partial f}{\partial y} = -2y$, is first positive (for $y < 0$) then negative (for $y > 0$) so the surface goes up then down, as y increases. That tells us that we have neither a maximum nor a minimum at the origin. We're lucky. Had they <u>both</u> given us curves which are concave up, and that would be the case if $\frac{\partial^2 f}{\partial x^2} > 0$ and $\frac{\partial^2 f}{\partial y^2} > 0$ at $x = y = 0$, then we <u>still</u> couldn't tell if we had a minimum. It only says that the surface curves up when you leave the origin in either the positive or negative x- or y-directions ... say east-west or north-south. But what about all the other directions?



S: Are there such surfaces?

P: Sure. Just imagine laying a bedsheet on the x-y plane and imagine the x axis going east-west and the y-axis going north-south. Then get four people standing somewhere along the positive and negative x- and y-axes: east and west and north and south of the origin. Now get them to lift the sheet. See? The surface you get will curve up when you leave the origin in either a north-south or east-west direction. Now let the entire x-y plane fall away (except the x- and y-axes of course, else we'll lose our four helpers). Now the sheet will still curve up in the direction of the axes, but will curve DOWN if you head, say, north-west or south-west and so on.

S: A picture is worth ...



S: So you're telling me that there isn't any test for a maximum or minimum. That I have to just look to see what happens to the surface when I go east-west or north-south. Is that it?

P: As a matter of fact there is a test ... a "second derivative test" ... which sometimes works, but I'd rather you didn't use it so I won't tell you what it is. What I want you to do is inspect the surface near the points where the partial derivatives are zero. See if it goes up in <u>every</u> direction (a minimum)... or down in <u>every</u> direction (a maximum) ... or up in some directions and down in others (a saddle point).

S: But how can I consider *every* direction, all at once?

P: You could start at (x_0, y_0), a place where $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, then you can move in some α-direction using the parametric equations for a line in 2-D, namely $x = x_0 + s\cos\alpha$, $y = y_0 + s\sin\alpha$, then see how f(x,y) behaves as s increases or decreases. If, for every angle α the value of f(x,y) increases, you've got a minimum. See?

S: Sort of ... but isn't that complicated?

P: Sort of, but only because of the terms $s\cos\alpha$ and $s\sin\alpha$. It's easier if you just put $x = x_0 + u$, $y = y_0 + v$ and see if every "u" and "v" will make z increase. But you only have to consider very small values of "u" and "v" and that makes it easier because you can neglect u^3 compared to u^2 and v^3 compared to v^2 and so on. In other words, you can approximate the values of your function, when u and v are really small, to see if the values increase or decrease.

S: Sounds tough.

Example:

Examine $z = xy + \frac{1}{x} + \frac{1}{y}$ for a minimum, in $x > 0$, $y > 0$. (There is no maximum because $z \to \infty$ as $x \to 0^+$)

Solution: We set $\frac{\partial z}{\partial x} = y - \frac{1}{x^2} = 0$ and $\frac{\partial z}{\partial y} = x - \frac{1}{y^2} = 0$, solve and obtain the solution $x = y = 1$, in which case $z = 3$ is the expected minimum value of z. Now substitute $x = 1 + u$, $y = 1 + v$ into z and get:

$z=(1+u)(1+v)+\frac{1}{1+u}+\frac{1}{1+v}$. Since we are only interested in the behaviour of z when u and v are very small (to see if z is always larger or smaller than 3), we can write $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \ldots$ and $\frac{1}{1+v} = 1 - v + v^2 - v^3 + \ldots$ (obtained by dividing into 1, or by recognizing the sum of an infinite geometric series) and get:

$$\boxed{z = 1 + u + v + uv + (1 - u + u^2 - u^3 + \ldots) + (1 - v + v^2 - v^3 + \ldots) = 3 + u^2 + uv + v^2 + \ldots}$$

where the "..." indicates quantities which are very much smaller than u^2, uv and v^2 (like u^3, for example). We conclude that, for u and v very small (meaning x and y are each near 1), z behaves like $3 + u^2 + uv + v^2$ so that it *does* have the value $z = 3$ when $u = 0$ and $v = 0$ (meaning $x = 1$, $y = 1$). The question is: "Is $u^2 + uv + v^2$ always positive?" If so, it means that $z > 3$ for all points near $u = 0$, $v = 0$ (meaning $x = 1$, $y = 1$ gives a minimum). The answer to this question is "Yes". In fact, $u^2 + uv + v^2$ can be rewritten as $(u + \frac{v}{2})^2 + \frac{3}{4}v^2$, where we've "completed the square". In this form we see $u^2 + uv + v^2$ as a sum of squares, so it <u>is</u> always positive, so z <u>does</u> have values larger than 3 when u and v are small (meaning x and y are each near 1).

We conclude that $x = y = 1$ provides a minimum value for $z = xy + \frac{1}{x} + \frac{1}{y}$.

S: It's not always that easy, right? I mean, it's not always that easy to see how z behaves when u and v are small.

P: Let's do another example. But first, remember the scheme:

(1) Set $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ and solve these two equations in two unknowns.

(2) Near each solution, (x_0, y_0), investigate the behaviour of f(x,y) by putting $x = x_0 + u$, $y = y_0 + v$ and retaining only the dominant terms (ignoring terms such as u^3 or u^2v etc. which are small in comparison to u^2 or uv or v^2).

(3) The expression for f will look like $\boxed{f(x_0, y_0) + A u^2 + B uv + C v^2}$ where everything else has been ignored.

(4) Investigate $A u^2 + B uv + C v^2$, completing the squares or otherwise, to see if it's always positive for every u and v ... in which case you've got a minimum since then $f(x,y) > f(x_0, y_0)$, or perhaps it's always negative (in which case you've got a maximum) or perhaps it's positive for some u,v and negative for others (so you have a SADDLE point).

Example:

Examine $f(x,y) = 5 - x^2 + 2x - y - \frac{y^4}{4}$ for extreme values (maximum or minimum).

Solution:

We set $\dfrac{\partial f}{\partial x} = -2x + 2 = 0$ and $\dfrac{\partial f}{\partial y} = -1 - y^3 = 0$ and get a single solution x = 1 , y = -1 and we wish to know if $f(1, -1) = \dfrac{25}{4}$ is a maximum or a minimum. Hence we put x = 1 + u, y = -1 + v (where u and v are the deviations of x and y from the values 1 and -1) and get $f(1+u, -1+v) = 5 - (1+u)^2 + 2(1+u) - (-1 + v) - \frac{1}{4}(-1 + v)^4$. As before, we can simplify this considerably, for small values of u and v, by ignoring "small terms", so we expand and write:

$$\boxed{f(1+u,-1+v) = 5 - (1+2u+u^2) + 2+2u +1-v - \frac{1}{4}(1-4v+6v^2-4v^3+v^4) = \frac{25}{4} - u^2 - \frac{3}{2}v^2 + v^3 - \frac{1}{4}v^4}$$

and now we ignore the v^3 and v^4 terms compared to the v^2 term (when u, v are small!) and get $f \approx \frac{25}{4} - u^2 - \frac{3}{2} v^2$ (when u, v are small!) so it's clear that f has values *smaller* than $\frac{25}{4}$ for all <u>small</u> values of u and v, so $\frac{25}{4}$ is a maximum. In this case we didn't even have to "complete the squares" to see that $-u^2 - \frac{3}{2} v^2$ was always negative. If, however, a "uv" term appears, then "completing the squares" almost essential.

S: Are you saying that $A u^2 + B uv + C v^2$ can never be negative, or maybe never positive?

P: Sure. Look at $u^2 + uv + v^2$, from an earlier example. It's <u>always</u> positive, regardless of what you plug in for u and v (except u = v = 0 of course, where it's zero). But $u^2 + 2uv - v^2$ is sometimes positive (try u = 2, v = 1) and sometimes negative (try u = 1, v = 2) and $-u^2 - uv - 2v^2$ is always negative. That means that when you put $x = x_0+u$, $y = y_0+v$ into f(x,y) and get something like $f(x,y) \approx f(x_0, y_0) + A u^2 + B uv + C v^2$ (after ignoring the smaller terms) then you can tell if $f(x,y) > f(x_0, y_0)$ or $f(x,y) < f(x_0, y_0)$ or neither of these just by investigating $A u^2 + B uv + C v^2$. Nice, eh?

S: Yeah, but what if you get terms with u and v, like maybe $f(x,y) \approx f(x_0, y_0) + A u + B v + ...$?

P: You won't, trust me. In fact, if you remember the equation of the tangent plane to z = f(x,y) at (x_0, y_0), it's $\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}$ and that's an approximation to the surface z = f(x,y) near $x = x_0$, $y = y_0$. However, the <u>exact</u> values of f(x,y) would <u>really</u> have to be written $\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + ...}$ where the "..." contained everything you ignored and would have stuff involving $(x - x_0)^2$ and $(y - y_0)^2$ and $(x - x_0)(y - y_0)$ which, of course, is just u^2 and v^2 and uv so we could write $z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\, u + \frac{\partial f}{\partial y}(x_0, y_0)\, v + A u^2 + B uv + C v^2 + ...$ where <u>now</u> the "..." includes stuff like u^3 and u^2v and even smaller terms, but if we now consider what happens when (x_0, y_0) is a point where both partial derivatives are zero, then, near such a point we'd have:

$\boxed{z = f(x_0, y_0) + A u^2 + B uv + C v^2 + ...}$ and you can see that there won't be any terms in "u" or "v". See? You won't believe this, but the above expression is really a Taylor series about (x_0, y_0) and we're neglecting the higher terms and retaining only the quadratic terms, as an approximation, just to see whether f(x,y) is always larger or smaller than $f(x_0, y_0)$ near the point (x_0, y_0). See? In fact, having said that, you might expect that the numbers A, B and C can be expressed in terms of the *second* derivatives of f(x,y) at the point (x_0, y_0) and you could generate a "second derivative test" for a maximum or minimum by investigating the sign or $A u^2 + B uv + C v^2$. See?

S: Hmmm.

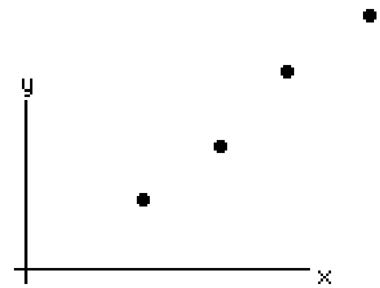
P: Don't worry, I only say these things to indicate how everything hangs together so nicely --- and in case you were wondering *"what about Tayor series for functions of several variables?"* because they're really useful in determining the local behaviour of functions and that's just what we need to investigate local maxima and minima. See?

S: Hmmm.

P: Well, it's time to move on ...

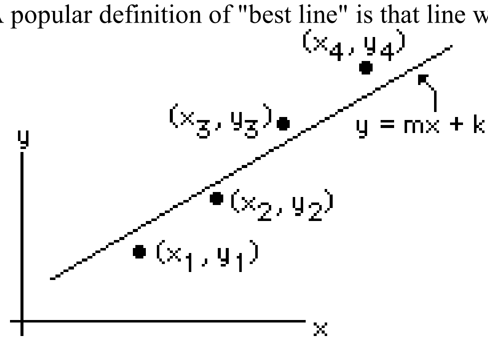
Least Squares Fit

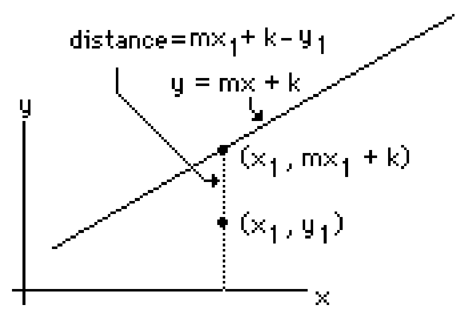
It often happens that we make several measurements, plot the points on a graph and notice that they almost lie on a straight line. We might, for example, measure a persons height as a function of his/her age and we'd obtain pairs of numbers (h_1, a_1) , (h_2, a_2) , etc. where the height was h_1 when the age was a_1, and so on. Or perhaps we'd measure the temperature of some chemical solution as a function of time, obtaining a temperature T_1 at time t_1, etc. Or maybe the cost of production as a function of the number of items manufactured, getting (C_1, N_1), (C_2, N_2) etc.



In each case we'd have a number of points: (x_1, y_1), (x_2, y_2), etc., and we'd plot them and they might look like they should lie on some straight line. What line? Clearly we want to find the "best" line, giving the "best" fit to the data points ... which, of course, brings us to what we mean by "best".

A popular definition of "best line" is that line which "minimizes the squared error" ... whatever that means.





Suppose we have points (x_1, y_1) etc. and we wish to find the "best" line: $y = mx + k$, meaning we must determine the "best" values for "m" and "k". At $x = x_1$, our data point has $y = y_1$ whereas the line has $y = m x_1 + k$ so the error is $(mx_1+k - y_1)$. At the next point, $x = x_2$ and our data point has $y = y_2$ but the line (whatever line we invent!) has $y = mx_2+k$ so the error here is $(mx_2+k - y_2)$. If we add all the errors we'd get $(mx_1+k - y_1) + (mx_2+k - y_2) + \ldots$ for as many data points as we have. Perhaps we can choose "m" and "k" so as to minimize this total error: it is, after all, a function of <u>two</u> variables (m,k) and we now know how to do that. Alas, this is a bad choice for our "best" line since some errors may be positive and some negative and they'd cancel even though the line was no where near the data points ... and we'd be very unhappy with a "minimum" error of $-\infty$ (and we couldn't do better than that minimum, could we?).

We need a better definition of "best" and it should be such that the error somehow gets larger as the line moves farther from the data points and we don't care whether a data point is below the line, giving a positive error $(mx_1+k - y_1)$, or above the line where this error would be negative. So we take the <u>squares</u> of the errors and that's what we'll minimize.

This definition of "best" has several nice features:

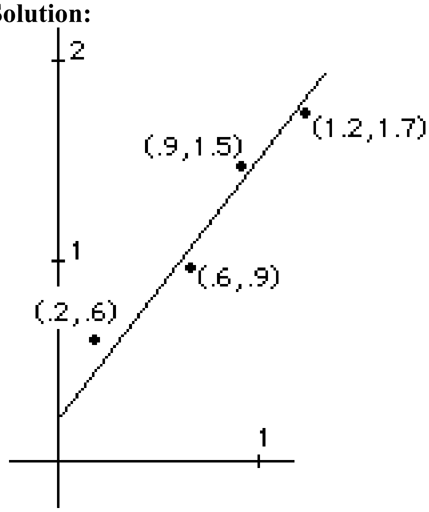
(1) The error is never negative; its minimum value is zero which is exactly what we'd like for the minimum error (meaning every point lies precisely on the line ... if we manage to reduce the error to zero!).

(2) It makes no difference whether a point is 0.1 units below or above the line: it makes a *positive* contribution to the error.

(3) Perhaps the most telling feature is that the math is easy! (We could also have chosen the sum of the *absolute values* of each error, but the math would be messy.)

Example: Determine the line of "least squared error" to fit the points (.2,.6), (.6,.9), (.9,1.5) and (1.2,1.7).

Solution:



If the line has equation $y = mx + k$, the four errors are : $m(.2)+k - .6$, $m(.6)+k - .9$, $m(.9)+k - 1.5$ and $m(1.2)+k - 1.7$ and the sum of their squares is:
$(.2m+k - .6)^2 + (.6m+k - .9)^2 + (.9m+k - 1.5)^2 + (1.2m+k - 1.7)^2$
which is a function of <u>two</u> variables m and k and we can set the <u>two</u> partial derivatives to zero. If we call the squared error f(m,k), then:

$$\frac{\partial f}{\partial m} = 2(.2m+k-.6)(.2)+2(.6m+k-.9)(.6)+2(.9m+k-1.5)(.9) +2(1.2m+k - 1.7)(1.2) = 0 \text{ and}$$

$$\frac{\partial f}{\partial k} = 2(.2m+k-.6)+2(.6m+k-.9)+2(.9m+k-1.5)+2(1.2m+k - 1.7) = 0$$

and this gives <u>two</u> equations in <u>two</u> unknowns, m and k.

We divide each equation through by 2, then rewrite them as:

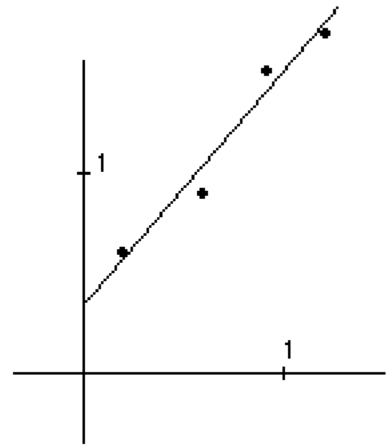
$$\boxed{[(.2)^2+(.6)^2+(.9)^2+(1.2)^2]\, m + [(.2)+(.6)+(.9)+(1.2)]\, k = [(.2)(.6)+(.6)(.9)+(.9)(1.5)+(1.2)(1.7)]} \quad \text{and}$$

$$\boxed{[(.2)+(.6)+(.9)+(1.2)]\, m + 4\, k = [(.6)+(.9)+(1.5)+(1.7)]}\ .$$

We've left things as they occur, without multiplying, so we can see what to do when there are 44 or 144 data points, instead of just 4. In general, for "n" data points, we'd get:

$$\boxed{[x_1{}^2+x_2{}^2+\ldots+x_n{}^2]\, m + [x_1+x_2+\ldots+x_n]\, k = [x_1y_1+x_2y_2+\ldots+x_ny_n]} \quad \text{and}$$

$$\boxed{[x_1+x_2+\ldots+x_n]\, m + n\, k = [y_1+y_2+\ldots+y_n]}\ .$$



Anyway, now that we see what the general equations are, let's proceed with our example.

We have:

$\boxed{(1)\quad 2.650\ m + 2.9\ k = 4.050}$ and

$\boxed{(2)\quad 2.9\ m + 4\ k = 4.7}$

so that, from (2), $k = \frac{4.7 - 2.9m}{4}$ which we substitute into (1) to get

$2.650\ m + 2.9\left(\frac{4.7 - 2.9m}{4}\right) = 4.050$ hence m = 1.174 and k = .324 and our "best" line is:

$\boxed{y = 1.174\ x + .324}$

The graph at the right shows how well the line fits the data.

Let's solve the "general" equations so we can use the results for future problems. To make things simpler, we'll use a short-hand SIGMA notation, writing

$\boxed{x_1{}^2+x_2{}^2+...+x_n{}^2=\sum x^2}$ $\boxed{x_1y_1+x_2y_2+...+x_ny_n=\sum xy}$ $\boxed{x_1+x_2+...+x_n=\sum x}$ and $\boxed{y_1+y_2+...+y_n=\sum y}$

so the general equations read:

$\boxed{(\sum x^2)\,m + (\sum x)\,k = (\sum xy)}$ and $\boxed{(\sum x)\,m + n\,k = \sum y}$.

We solve for m and k and get:

the "least squares" straight line fit is y = mx + k, where

$$m = \frac{n\sum xy - \sum x\sum y}{n\sum x^2 - (\sum x)^2} \quad \text{and} \quad k = \frac{\sum x^2\sum y - \sum x\sum xy}{n\sum x^2 - (\sum x)^2}$$

For our example above, we might make up a table like so:

x^2	x	y	xy
.04	.2	.6	.12
.36	.6	.9	.54
.81	.9	1.5	1.35
1.44	1.2	1.7	2.04
$\sum x^2 = 2.65$	$\sum x = 2.9$	$\sum y = 4.7$	$\sum xy = 4.05$

where the middle two columns are the given data points.

Then $m = \frac{n\sum xy - \sum x\sum y}{n\sum x^2 - (\sum x)^2} = \frac{4(4.05) - (2.9)(4.7)}{4(2.65) - (2.9)^2} = 1.174$ and

$k = \frac{\sum x^2\sum y - \sum x\sum xy}{n\sum x^2 - (\sum x)^2} = \frac{(2.65)(4.7) - (2.9)(4.05)}{4(2.65) - (2.9)^2} = .324$

S: Whew! I hope you don't expect me to memorize all that!

P: No, but you should be able to generate those equations on your own. After all, it's just squaring each error, adding them all, then a couple of partial derivatives then solving just two equations in two unknowns. Besides, look how nice the result is. If the data points are measuring temperature at various times ... that is, y_1 and y_2 etc. are measured in *degrees* ... and x_1, x_2 etc. are in *minutes* so your measuring temperatures every few minutes ... then each term in the equation $y = mx + k$ must have the same dimensions, namely *degrees* (the same as y), so "k" is in *degrees* and so is "mx" and that means that "m" is measured in *degrees/minute* (it is, after all, $m = \frac{dy}{dx}$ = the rate of change of y with respect to x). Now look at each $\sum$ in the equations above:

$\sum x^2$ is in *minutes*^{2}, $\sum x$ is in *minutes*, $\sum y$ is in *degrees* and $\sum xy$ is in *minute-degrees*.

Then $k = \frac{\sum x^2 \sum y - \sum x \sum xy}{n \sum x^2 - (\sum x)^2}$ has the dimensions of $\frac{\text{minutes}^2\text{-degrees}}{\text{minutes}^2}$ = degrees, as it should, and

$m = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$ has the dimensions of $\frac{\text{minutes-degrees}}{\text{minutes}^2} = \frac{\text{degrees}}{\text{minute}}$, as it should.

S: But what if you only have two points, like (x_1, y_1) and (x_2, y_2). Then there __is__ a line which goes right through those points. Will the "least squares " line do that ... I mean go through those two points?

P: Yes. You'll get $m = \frac{y_2 - y_1}{x_2 - x_1}$ which is the appropriate slope, and you also get the appropriate k. In fact, the line you get can be written $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ as you'd expect.

S: But you'd be in big trouble if the denominator is zero, right? I mean, suppose you had points which made $n \sum x^2 - (\sum x)^2 = 0$. Then m has a zero denominator and so does k. Right?

P: It won't happen. In fact the denominator is always greater than zero. In fact, if you look at the equations for "m" and "k" and divide numerator and denominator by n^2 you see things which look like averages. For example, you'll see $\frac{1}{n} \sum x = \frac{x_1 + x_2 + ... + x_n}{n}$ which is the average of the x-values, and $\frac{\sum x^2}{n}$ which is the average value of x^2 and $\frac{\sum xy}{n}$, the average value of xy and so on. If we give them names, like $\frac{\sum x}{n} = <x>$ and $\frac{\sum x^2}{n} = <x^2>$ and $\frac{\sum xy}{n} = <xy>$ and so on, then after we divide numerator and denominator by n^2 we can write:

$$\boxed{m = \frac{<xy> - <x><y>}{<x^2> - <x>^2} \text{ and } k = \frac{<x^2><y> - <x><xy>}{<x^2> - <x>^2}}$$

Now you won't believe this but $<x^2> - <x>^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$ is nobody else but $\frac{\sum (x - <x>)^2}{n}$.

That is, $\frac{x_1^2 + x_2^2 + ...}{n} - \left(\frac{x_1 + x_2 + ...}{n}\right)^2 = \frac{(x_1 - <x>)^2 + (x_2 - <x>)^2 + ...}{n}$ so it's a sum of squares so it's always positive ... and never zero. I leave that as an exercise for the student ... and guess who's the student?

S: But what if the data points __don't__ look like they lie on a line?

P: Then assume some other function instead of $y = mx + k$. For example, you might expect populations to grow exponentially, so you'd assume $y = K e^{bx}$ where K and b are constants. Maybe they look like they lie on a parabola, so you'd assume $y = a + bx^2$ where "a" and "b" are constants. In each case you'd find the sum of the squares of the errors and it would be a function of the constants K and b, or a and b, or whatever. Then you'd set the partial derivatives to zero and solve for these constants. See? There will be as many equations as there are unknown constants. Sometimes they'd be easy to solve, as was the case for the least squares line ... sometimes they'd be difficult.

a difficult Example:

Data points (.1, 10), (.2, 16), (.3, 25) are given. Find equations for the least squares exponential fit: $y = K e^{bx}$.

Solution:

The sum of the squared errors is $E(K,b) = (Ke^{.1b} - 10)^2 + (K e^{.2b} - 16)^2 + (K e^{.3b} - 25)^2$, a function of the two variables K and b. We set

$$\boxed{\frac{\partial E}{\partial K} = 2(Ke^{.1b} - 10)e^{.1b} + 2(K e^{.2b} - 16)e^{.2b} + 2(K e^{.3b} - 25)e^{.3b} = 0} \text{ and}$$

$$\boxed{\frac{\partial E}{\partial b} = 2(Ke^{.1b} - 10).1K + 2(K e^{.2b} - 16).2K + 2(K e^{.3b} - 25).3K = 0} \ .$$

These terrible equations can be simplified somewhat by letting $e^{.1b} = x$ so $e^{.2b} = x^2$ and $e^{.3b} = x^3$ (so we're lucky the x-values were equally spaced!). Then we still get two terrible equations

$\boxed{(Kx-10) + (Kx^3-16x) + (Kx^5-25x^2) = 0}$ and $\boxed{(.1Kx-1) + (.2Kx^2-3.2) + (.3Kx^3-7.5) = 0}$.

These look like $\boxed{1: p_5 K + p_2 = 0}$ and $\boxed{2: p_3 K + p_0 = 0}$ where p_5 is a polynomial of degree 5, p_2 is a poly of degree 2, and so on. To eliminate K, multiply (1) by p_3 and (2) by p_5 and subtract, giving $p_3 p_2 - p_5 p_0 = 0$, a polynomial of degree 5 in the variable $x = e^{.1b}$. To solve this we could use Newton's Method.

S: That's awful!
P: Why? Because it's a lot of work or because you don't understand what's happening?
S: Because it's a lot of work.
P: I'm glad you said that. But remember, if you want to apply mathematics to real-world problems you'll often need some functions which describe, say, the profit as a function of number items manufactured and time of year --- and so on. Where do you think these functions come from? Nobody will walk up to you and say: "Please maximize my profit: $P(x,y) = 5 - x^2 + 2x - y - \frac{y^4}{4}$." You only find these ridiculous problems in text books.
S: Like this one?
P: Well ... yes. But in real life, you'll need to derive some functions, often from observations --- the *data points* we've considered above --- and that often means finding the "best fit" --- and that may take time but you can have some faith in the function because it's the best fit to actual observations.
....
Anyway, you may be interested to know that this course is over.
S: Hmmm ... too bad.

SOLUTIONS TO "ASSORTED PROBLEMS"

1. At the point of inflection, t = 2 and F(2) = K/2, hence K = 2 F(2) = 2 (10,000) = 20,000 fish. To simplify, let $y = \frac{F}{K}$ so the growth is described by: $y(t) = \dfrac{1}{1+\left(\frac{1-y_0}{y_0}\right)e^{-rt}}$ where $y_0 = \frac{F(0)}{K}$. We have the following :

   When t = 2 then $y(2) = \frac{F(2)}{K} = \frac{1}{2} = \dfrac{1}{1+\left(\frac{1-y_0}{y_0}\right)e^{-2r}}$ so $\boxed{\left(\frac{1-y_0}{y_0}\right)e^{-2r} = 1}$. When t = 3 then

   $y(3) = \frac{F(3)}{K} = \frac{3}{4} = \dfrac{1}{1+\left(\frac{1-y_0}{y_0}\right)e^{-3r}}$ so $\boxed{\left(\frac{1-y_0}{y_0}\right)e^{-3r} = \frac{1}{3}}$. We now have two equations in two unkowns.

   Solve for $e^r = 3$, hence $r = \ln 3$, and $y_0 = \frac{1}{10}$, hence $F(0) = K\, y_0 = 20{,}000(1/10) = 2{,}000$ fish (at t=0).

2. (a) The DE is LINEAR, so write it in "standard form": $\frac{dy}{dx} + \frac{3}{x}\, y = -\frac{2}{x^2}$. Now evaluate $\int \frac{3}{x}\, dx = 2 \ln x$ and multiply the DE by the "integrating factor" $e^{3 \ln x} = e^{\ln x^3} = x^3$ and get $x^3 \frac{dy}{dx} + 3x^2 y = -2x^3$ and the left-side is exactly $\frac{d}{dx}(x^3 y)$ ... (meaning that we've calculated the integrating factor correctly!). The DE now reads: $\frac{d}{dx}(x^3 y) = -2x^3$ so $x^3 y = -\frac{1}{2} x^4 + C$ ... integrating both sides. The solutions are: $y = -\frac{x}{2} + \frac{C}{x^3}$.

   (b) The DE is LINEAR so we multiply by $\exp(\int 2x\, dx) = e^{x^2}$ and get :

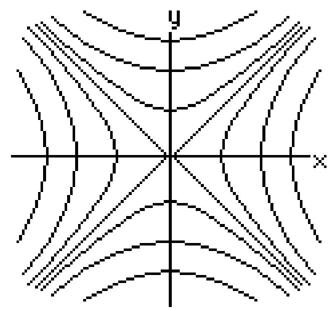
$e^{x^2} \frac{dy}{dx} + 2\,x\,e^{x^2}\,y = x\,e^{x^2}$ or

Error!e^{x^2} + C so the solutions are: y = **Error!**+ C e^{-x^2}.

Note: The DE is also "separable: write $\frac{dy}{dx} + 2\,x\,y = x$ as $\frac{dy}{1 - 2y} = x\,dx$ and integrate each side, giving: $-\frac{1}{2}\,ln\,|\,1 - 2y\,| = \frac{x^2}{2} + C$ which we could (if we really wanted to) solve for $y = \frac{1}{2} + K\ e^{-x^2}$ (as before).

3. Differentiate the DE to get $10\,x + 4y + 4x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{2y + 5x}{2y + 2x}$. The Orthogonal Trajectories will satisfy $\frac{dy}{dx} = \frac{2y + 2x}{2y + 5x}$ (the negative reciprocal) .

4. f(x,y) = constant gives $1 - x^2 + y^2$ = constant which is the same as: $y^2 - x^2$ = constant, a family of hyperbolas (diagram at right).



5. If $z = x^2+xy$ then, when x = 1, y = 2, we have z = 3 and $\frac{\partial z}{\partial x} = 2x + y = 4$ and $\frac{\partial z}{\partial y} = x = 1$. The tangent plane is then (4)(x-1) + (1)(y-2) + (-1)(z-3) = 0, or 4x + y - z = 3 ... where we've used the standard equation $\frac{\partial f}{\partial x}(a,b)\,(x - a) + \frac{\partial f}{\partial y}(a,b)\,(y - b) + (-1)\,(z - f(a,b)) = 0$ for the tangent plane to z = f(x,y) at (a,b).

6.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^n - n^\pi}$. Discussion: the exponential function π^n is much larger than n^π, when n is large (... think of 3^x versus x^3) so the series looks like $\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^n}$ (when n is large) and this converges ... even if we sum the absolute values (... think of $\sum \frac{1}{\pi^n}$ which behaves like $\sum e^{-n}$ where the terms will certainly go to zero quickly enough for convergence) ... so we omit the alternating series test and consider, directly, whether the original series converges absolutely ... so we test $\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$, comparing it to $\sum_{n=1}^{\infty} \frac{1}{\pi^n}$. In order effect this comparison we need the ratio of terms to approach "1", so consider $\frac{1/\pi^n}{1/(\pi^n - n^\pi)} = 1 - \frac{n^\pi}{\pi^n}$. To have a limit of "1" we need $\lim_{n\varnothing\infty} \frac{n^\pi}{\pi^n} = 0$. This has the form $\frac{\infty}{\infty}$ so we can use l'Hopital's rule: $\lim_{n\varnothing\infty} \frac{n^\pi}{\pi^n} =$

Error!. We continue applying l'Hopital's rule until we get **Error!**= **Error!**and now we have the form **Error!**so the limit is "0" ... and we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$ will converge (or diverge) depending upon whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n}$ converges (or diverges), but this is a geometric series with common ratio $\frac{1}{\pi}$ which is less than 1 so it converges,

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$ converges, so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^n - n^\pi}$ converges absolutely.

(b) $\displaystyle\sum_{i=1}^{\infty} e^i$. Discussion: we write this as $e + e^2 + e^3 + \ldots$ and recognize a geometric series with common ratio "e" which is larger than "1", so our series diverges. (Write out a few terms ... it's often useful.)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2 + 1}$. Discussion: for large "n", the terms look like $\frac{n^2}{2n^2} = \frac{1}{2}$ hence they don't have a limit of zero so

the series diverges. To get "full marks", write $\displaystyle\lim_{n \to \infty} \frac{n^2}{2n^2+1} = \lim_{n \to \infty} \frac{1}{2+1/n^2} = \frac{1}{2} \neq 0$ hence series diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\, e^n}$. Discussion: the terms of this series are even smaller than those of $\sum \frac{1}{e^n}$ which converges (it's a

geometric series with common ratio $\frac{1}{e} < 1$), hence the partial sums are increasing and bounded by $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

hence the partial sums converge ... and that's the definition of convergence for the original series (and since every term is positive, it clearly converges absolutely!).

(e) $\displaystyle\sum_{i=1}^{\infty} \frac{i^2}{1+i\sqrt{i}}$. Discussion: for large "i" the terms look like $\frac{i^2}{i\sqrt{i}} = \sqrt{i}$ so they don't have a limit of zero. Consider,

then, $\displaystyle\lim_{i \to \infty} \frac{i^2}{1+i\sqrt{i}} = \lim_{i \to \infty} \frac{\sqrt{i}}{i^{-3/2}+1}$ which has the form $\frac{\infty}{1}$, so the limit is ∞, so the series diverges.

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin n\pi}{\mathit{ln}\ n}$. The terms are $\frac{\sin 2\pi}{\mathit{ln}\ 2} + \frac{\sin 3\pi}{\mathit{ln}\ 3} + \ldots = 0 + 0 + \ldots$ so the series converges to 0. (It pays to write out a few terms if you don't have a feel for what the terms look like!)

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$. The terms are $-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + - \ldots$ so it's the alternating harmonic series ... and since the terms

decrease to zero, the series converges ... but <u>not</u> absolutely because $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ is the divergent harmonic series.

7. (a) $\int \frac{dy}{e^y} = \int x\,dx + C$ gives: $-e^{-y} = \frac{x^2}{2} + C$ or $y = -ln\left(K - \frac{x^2}{2}\right)$ (where K = - C)

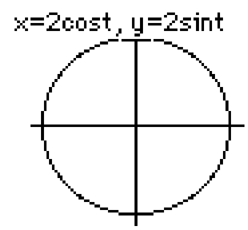
(b) $\int \frac{dy}{1-y^2} = \frac{1}{2}\int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy = \int dx$ gives $ln\left|\frac{1+y}{1-y}\right| = 2x+C$ or $y = \frac{Ke^{2x}-1}{Ke^{2x}+1}$ (where $K = \pm e^C$)

(c) $\int \frac{dy}{y-1} = \int x\,dx$ gives $ln\,|y-1| = \frac{x^2}{2} + C$ or $y = 1 + K\,e^{x^2/2}$ (where $K = \pm e^C$)

(d) $\int \frac{dy}{y} = 2\int \frac{dx}{x}$ gives $ln\,|y| = 2\,ln\,|x| + C = ln\,x^2 + C$ or $|y| = e^C x^2$ or $y = K x^2$ (where $K = \pm e^C$)

8. (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -\cot t = -1$ when $t = \pi/4$ and

$L = \int_0^{2\pi}\sqrt{4\sin^2 t + 4\cos^2 t}\,dt = 2\int_0^{2\pi} dt = 4\pi$ (the circumference of a circle of radius 2)

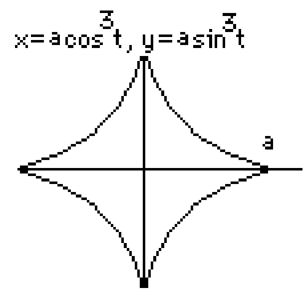


(b) $\frac{dy}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t = -1$ at $t = \pi/4$

and $L = 4\int_0^{\pi/2}\sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt$

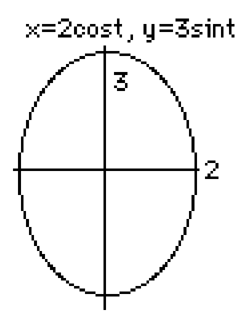
$= 12a\int_0^{\pi/2}\sqrt{\cos^2 t\sin^2 t\,(\cos^2 t + \sin^2 t)}\,dt = 12a\int_0^{\pi/2}|\cos t\sin t|\,dt$

$= 12a\int_0^{\pi/2}\cos t\sin t\,dt = 12\,a\left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = 6$



(c) $\frac{dy}{dx} = \frac{3\cos t}{-2\sin t} = -\frac{3}{2}\cot t = 0$ at $t = \pi/2$

and $L = \int_0^{2\pi}\sqrt{4\sin^2 t + 9\cos^2 t}\,dt$ (a so-called *elliptic integral*)



9.

(a) $f(a) = \sin a = 0$; $f'(a) = \cos a = 1$; $f''(a) = -\sin a = 0$; $f'''(a) = -\cos a = -1$; $f^{(4)}(a) = \sin a = 0$; $f^{(5)}(a) = \cos a = 1$; $f^{(6)}(a) = -\sin a = 0$

Then $P_6(x) = f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2!+f'''(a)(x-a)^3/3!+f^{(4)}(a)(x-a)^4/4!+f^{(5)}(a)(x-a)^5/5!+ f^{(6)}(a)(x-a)^6/6!$

gives $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

(b) Use $f^{(n)}(a)$ (from part (a)), evaluate at $a = \pi/2$, and get: 1, 0, -1, 0, 1, 0, -1

Then $\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + \ldots$

Note: put $x = t + \frac{\pi}{2}$ and get $\sin(t + \frac{\pi}{2}) = \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$

(c) $f^{(n)}(a) = e^a = e$ (for $a = 1$) so we get:

$$e^x = e + e\,(x-1) + e\frac{(x-1)^2}{2!} + e\frac{(x-1)^3}{3!} + e\frac{(x-1)^4}{4!} + \ldots$$

Note: this gives $e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}$ + etc. (i.e. $e^t = 1+t+\frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$ with $t=x-1$

(d) $f(a) = \ln(1+a) = 0$; $f'(a) = 1/(1+a) = 1$; $f''(a) = -1/(1+a)^2 = -1$; $f'''(a) = 2/(1+a)^3 = 2!$; $f^{(4)}(a) = -3!/(1+a)^4 = -3!$;

Then $f(x) = f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + f^{(4)}(a)\frac{(x-a)^4}{4!} + \ldots$

gives $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

Note: $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \frac{t^4}{1+t}$ (an identity which follows from the sum of a geometric series) so

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = \int_0^x (1 - t + t^2 - t^3 + \ldots dt) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ (very nice!)}$$

(e) $f(a) = \tan a = 0$; $f'(a) = \sec^2 a = 1$; $f''(a) = 2\sec^2 a \tan a = 0$; $f'''(a) = 2\sec^2 a\,(2\tan^2 a+\sec^2 a) = 2$;
$f^{(4)}(a) = 8\sec^2 a \tan a\,(\tan^2 a+2\sec^2 a) = 0$; $f^{(5)}(a) = 8\sec^2 a\,(2\tan^4 a+11\sec^2 a\tan^2 a+2\sec^4 a) = 16$;

which gives $\tan x = x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \ldots$

Note: dividing $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ by $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ gives $\tan x \approx x + \frac{x^3}{3} + \frac{2}{15}x^5$,
the same polynomial approximation as we obtained above.

(f) $f(a) = \arcsin a = 0$; $f'(a) = 1/\sqrt{1-a^2} = 1$; $f''(a) = a/(1-a^2)^{3/2} = 0$; $f'''(a) = (1+2a^2)/(1-a^2)^{5/2} = 1$;

giving $\arcsin x = x + \frac{x^3}{3!} + \ldots$

(g) $f(a)=\sqrt{1+a} = 1$; $f'(a) = 1/2\sqrt{1+a} = 1/2$; $f''(a) = -1/4(1+a)^{3/2} = -1/4$; $f'''(a) = 3/8(1+a)^{5/2} = 3/8$;

giving $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} + \ldots$

Note: the terms in the polynomial approximation give the so-called "Binomial Expansion" of $(1+x)^{1/2}$.

(h) $f(a) = \sinh a = 0$; $f'(a) = \cosh a = 1$; $f''(a) = \sinh a = 0$; $f'''(a) = \cosh a = 1$; $f^{(4)}(a) = \sinh a = 0$; $f^{(5)}(a) = \cosh a = 1$;

so we get $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

Note: $\sinh x = \frac{e^x - e^{-x}}{2}$ so the approximation can also be obtained from the Taylor polys for e^x and e^{-x}.

(i) $f(a) = \cosh a = 1$; $f'(a) = \sinh a = 0$; $f''(a) = \cosh a = 1$; $f'''(a) = \sinh a = 0$; $f^{(4)}(a) = \cosh a = 1$; $f^{(5)}(a) = \sinh a = 0$

so we get $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

Note: Since $\cosh x = \frac{e^x + e^{-x}}{2}$ the approximation can also be obtained from the Taylor polys for e^x and e^{-x}.

(j) $f(a) = (1+a)/(1-a) = 1$; $f'(a) = 2/(1-a)^2 = 2$; $f''(a) = 4/(1-a)^3 = 4$; $f'''(a) = 12/(1-a)^4 = 12$; so:

$$\frac{1+x}{1-x} = 1 + 2x + 4\frac{x^2}{2!} + 12\frac{x^3}{3!} + \ldots = 1 + 2x + 2x^2 + 2x^3 + \ldots$$

Note that $\frac{1+x}{1-x} = \frac{2}{1-x} - 1 = 2(1+x+x^2+x^3+\ldots) - 1$ where $\frac{1}{1-x}$ is the sum of the geometric series $1+x+x^2+x^3+\ldots$

10.

(a) The Taylor series about x = 0 is: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - ....$ The Ratio Test gives $\rho = \lim_{n \to \infty} \frac{n}{n+1} |x| = |x|$

hence the series converges for $-1 < x < 1$ and diverges for $|x| > 1$. At x = 1 get $1 - \frac{1}{2} + \frac{1}{3} + -$

which converges (the alternating series test: the terms *decrease to zero*). At x = -1 get $-1 - \frac{1}{2} - \frac{1}{3} - ...$

the ($-^{ve}$) harmonic series ... hence diverges. Hence, interval of convergence is $-1 < x \leq 1$.

(b) Put x = t-1 in series for part (a) and get $\ln t = (t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - + ...$ hence the Taylor series for *ln* x,

about x = 1, is $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + - ...$ The Ratio Test gives $\rho = \lim_{n \to \infty} \frac{n}{n+1} |x-1| = |x-1|$ hence series converges for $|x-1|<1$, or $-1<x-1<1$, or $0<x<2$ (and diverges for $|x-1|> 1$, or $x>1$ and $x<-1$).

For x = 0 get $-1 - \frac{1}{2} - \frac{1}{3} - ...$ which diverges. For x = 2 get $1 - \frac{1}{2} + \frac{1}{3} + - ...$ which converges.

Interval of convergence is $0 < x \leq 2$.

(c) The Taylor series for e^x is $1+x+\frac{x^2}{2!}\frac{x^3}{3!} + ...$ Replace x with 2x and get the Taylor series for e^{2x}, namely

$1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + ...$ The Ratio Test gives $\rho = \lim_{n \to \infty} \frac{2x}{n+1} = 0$ for all x, hence interval of convergence is $-\infty < x < \infty$. (i.e series converges for all x.)

(d) Taylor series for sin x about $x = \frac{\pi}{4}$ is : $\sin\frac{\pi}{4} + \cos\frac{\pi}{4}(x-\frac{\pi}{4}) - \sin\frac{\pi}{4}(x-\frac{\pi}{4})^2/2! - \cos\frac{\pi}{4}(x-\frac{\pi}{4})^3/3! + ...$

The Ratio Test gives $\rho = \lim_{n \to \infty} \frac{|x-\frac{\pi}{4}|}{n+1} = 0$ for all x hence the interval of convergence is $-\infty < x < \infty$.

(e) The Taylor series for $\frac{1}{1-x}$ is $1 + x + x^2 + x^3 + ...$ (the sum of the geometric series is $\frac{1}{1-x}$, or you may use Taylors formula). The Ratio Test gives $\rho = \lim_{n \to \infty} |\frac{x^{n+1}}{x^n}| = |x|$ hence the series converges for $-1 < x < 1$ which is the interval of convergence (clearly the series diverges for $x = \pm 1$).

(f) The Taylor series for $\sinh x = \frac{1}{2}(e^x + e^{-x})$ is $\frac{1}{2}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + - ...\right)\right)$

or $x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$ and the Ratio Test gives $\rho = \lim_{n \to \infty} \frac{x^2}{2n(2n+1)} = 0$ for all x hence interval is $-\infty < x < \infty$.

11. $\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + ...) = a_0 + a_1 x + a_2 x^2 + ...$ gives

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ... = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

hence $a_1 = a_0$ and $2a_2 = a_1$ so $a_2 = \frac{a_0}{2}$ and $3a_3 = a_2$ so $a_3 = \frac{a_0}{2(3)} = \frac{a_0}{3!}$ and $4a_4 = a_3$ so $a_4 = \frac{a_0}{4!}$ etc. etc.

The power series $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ then becomes: $a_0 + a_0 x + a_0 \frac{x^2}{2!} + a_0 \frac{x^3}{3!} + a_0 \frac{x^4}{4!} + ...$

(namely $a_0 e^x$ ... which should be no surprise!)

12. Put x = 1 into $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ and get $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + ...$ so error in using just five terms, namely $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ is $\frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + ... = \frac{1}{5!}(1 + \frac{1}{6} + \frac{1}{6.7} + \frac{1}{6.7.8} + ...)$

which is less than $\frac{1}{5!}\left(1+\frac{1}{6}+\frac{1}{6.6}+\frac{1}{.6.6}+\ldots\right) = \frac{1}{5!}\frac{1}{1-\frac{1}{6}} = .01$ (where we recognize the geometric series!).

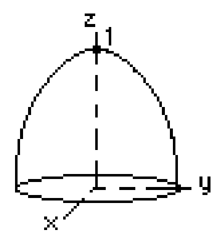
13. $1+x+x^2+x^3+\ldots = \frac{1}{1-x}$ (the sum of the geometric series) and differentiating each side gives

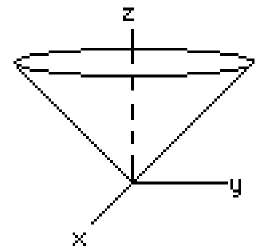
$1+2x+3x^2+4x^3+\ldots=\frac{1}{(1-x)^2}$ and multiplying each side by x gives

$x+2x^2+3x^3+4x^4+\ldots=\frac{x}{(1-x)^2}$ and we have summed this series!

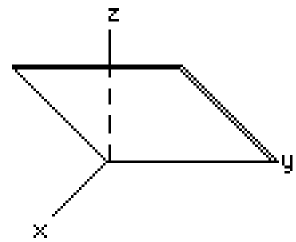
Hence $<E> = \frac{h\omega}{2\pi}\frac{\frac{x}{(1-x)^2}}{\frac{1}{1-x}} = \frac{h\omega}{2\pi}\frac{1}{\frac{1}{x}-1} = \frac{\frac{h\omega}{2\pi}}{e^{h\omega/2\pi kT}-1}$.

14. (a) Revolving the parabola $z = 1 - x^2$ about the z-axis generates the paraboloid $z = 1 - x^2 - y^2$ (replacing x^2 by $x^2 + y^2$).

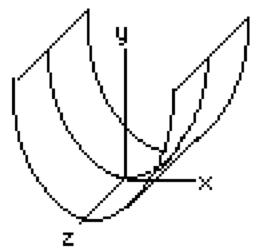




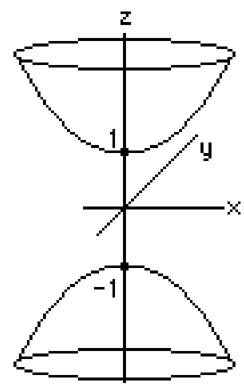
(b) The cone $z = \sqrt{x^2 + y^2}$ is obtained by revolving $z = \sqrt{x^2} = |x|$ about the z-axis (replacing x^2 by $x^2 + y^2$).



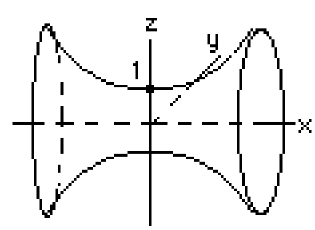
(c) $z = x$ is a line in the x-z plane and, moved parallel to the y-axis, becomes the plane $z = x$. (It can also be called a *cylinder* ?!).



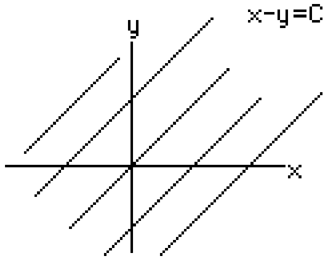
(d) The parabola $y = x^2$, in the x-y plane, when moved parallel to the z-axis, generates the (parabolic) cylinder $y = x^2$.

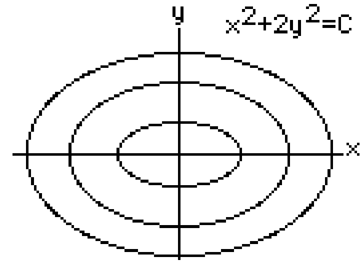


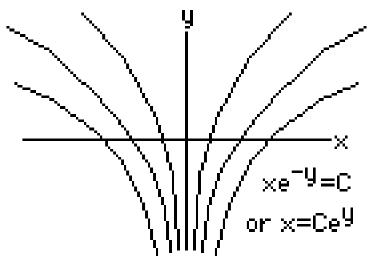
(e) The hyperbola $z^2 = 1 + x^2$ (or $z^2 - x^2 = 1$), when revolved about the z-axis, yields the hyperboloid (of 2 sheets) described by $z^2 = 1 + x^2 + y^2$ (replacing x^2 by $x^2 + y^2$).

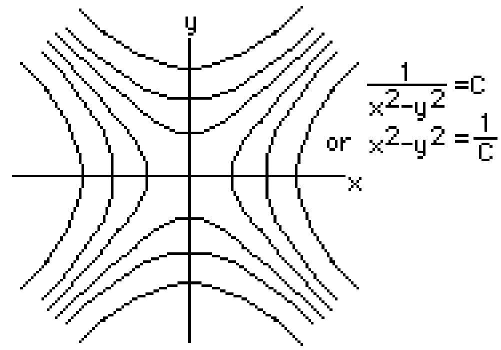


(f) The hyperbola $z^2 = 1 + x^2$ (or $z^2 - x^2 = 1$) , when revolved about the x-axis, yields the hyperboloid (of 1 sheet) described by $z^2 = 1 + x^2 - y^2$ (replacing z^2 by $z^2 + y^2$).

15. (a)


(b)


(c)


(d)


16. (a) $\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = -\frac{1}{2}$ at (-1,1) and $\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = -\frac{1}{2}$ at (-1,1)

hence equation of the tangent plane is: $z = -\frac{\pi}{4} - \frac{1}{2}(x+1) - \frac{1}{2}(y-1)$

(b) $\frac{\partial z}{\partial x} = -\frac{1}{2}\left(x^2+y^2\right)^{-3/2}(2x) = \frac{3}{125}$ at (-3,4) and $\frac{\partial z}{\partial y} = -\frac{1}{2}\left(x^2+y^2\right)^{-3/2}(2y) = -\frac{4}{125}$ at (-3,4)

hence equation of the tangent plane is: $z = \frac{1}{5} + \frac{3}{125}(x+3) - \frac{4}{125}(y-4)$

(c) $\frac{\partial z}{\partial x} = \frac{2y(y^2-x^2)}{(x^2+y^2)^2} = 1$ at (0,2) and $\frac{\partial z}{\partial y} = \frac{2x(x^2-y^2)}{(x^2+y^2)^2} = 0$ at (0,2)

hence equation of the tangent plane is: $z = 0 + 1(x-0) + 0(y-2)$ or $z = x$

(d) $\frac{\partial z}{\partial x} = \frac{1}{1+e^{xy}}\, e^{xy}\, y = \frac{1}{2}$ at (0,1) and $\frac{\partial z}{\partial y} = \frac{1}{1+e^{xy}}\, e^{xy}\, x = 0$ at (0,1)

hence equation of the tangent plane is: $z = \ln 2 + \frac{1}{2}(x-0) + 0(y-1)$ or $z = \ln 2 + \frac{1}{2}x$

17. (a) $\frac{\partial u}{\partial t} = -2e^{-2t}\sin x$ and $\frac{\partial^2 u}{\partial x^2} = -e^{-2t}\sin x$ hence $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with k = 2

(b) To simplify the computation of $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ we use logarithmic differentiation:

Write $\ln u = -\frac{x^2}{4t} - \frac{1}{2}\ln t$ so $\frac{1}{u}\frac{\partial u}{\partial t} = \frac{x^2}{4t^2} - \frac{1}{2t}$ so $\frac{\partial u}{\partial t} = u\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right) = \frac{u}{2t}\left(\frac{x^2}{2t} - 1\right)$

Also $\frac{1}{u}\frac{\partial u}{\partial x} = \frac{-x}{2t}$ so $\frac{\partial u}{\partial x} = -\frac{1}{2t}\,u\,x$. Then $\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t}\left(u + x\frac{\partial u}{\partial x}\right) = -\frac{1}{2t}\left(u - \frac{1}{2t}u\,x^2\right) = \frac{\partial u}{\partial t}$, so k = 1.

18. (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \sin 2y \cos 3t - 4\sin x \sin 2y \cos 3t = -5\sin x \sin 2y \cos 3t$

and $\frac{\partial^2 u}{\partial t^2} = -9 \sin x \sin 2y \cos 3t$ hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{9}{5}\frac{\partial^2 u}{\partial t^2}$ hence $\frac{1}{c^2} = \frac{9}{5}$ so $c = \frac{\sqrt{5}}{3}$.

(b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 9\ e^{3x+4y+5t} + 16\ e^{3x+4y+5t} = 25\ e^{3x+4y+5t}$

and $\frac{\partial^2 u}{\partial t^2} = 25\ e^{3x+4y+5t}$ hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$ and c = 1.

19. (a) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x\, e^t + 2y \cos t = 2(1)(1) + 2(0)(1) = 2$ when $t = 0$

(b) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2x\,(-\sin t) + 2y\,(\cos t) = 2(-1)(0) + 2(0)(-1) = 0$ when $t = \pi$

(c) $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\frac{du}{dz} + \frac{\partial w}{\partial v}\frac{dv}{dz} = (2u+v)\frac{1}{z} + u\,(2z) = (2.0+1)(1) + (0)(2.1) = 1$ when $z = 1$

(d) $\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dz} = (\text{Arctan } v)(\cos x) + \frac{u}{1+v^2}\sec^2 x = (\frac{\pi}{4})(\frac{1}{\sqrt{2}}) + \frac{1/\sqrt{2}}{1+1^2}(\sqrt{2})^2 = \frac{\pi}{4\sqrt{2}} + \frac{\sqrt{2}}{2}$

(e) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = 2x \cos t + 2y\,(-\sin t) + 2z\,(1) = 2(1)(0) + 2(0)(-1) + 2\frac{\pi}{2} = \pi$

20. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (-2x)(2) + (-4y)(2t) = (-2(7))(2) + (-4(4))(2(2)) = -28 - 64 = -92$ when $t = 2$

21. (a) $\frac{d}{dx}$ gives $2x + 2y\frac{dy}{dx} = 0$ so (solving) $\frac{dy}{dx} = -\frac{x}{y}$ and $-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2x}{2y} = -\frac{x}{y}$ as well.

(b) $\frac{d}{dx}$ gives $2xy + x^2\frac{dy}{dx} + \cos(xy)\,(x\frac{dy}{dx} + y) - 1 = 0$ so $\frac{dy}{dx} = -\frac{2xy + y\cos(xy) - 1}{x^2 + x\cos(xy)} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

(c) $\frac{d}{dx}$ gives $\frac{1}{1+(x^2+y^2)^2}(2x + 2y\frac{dy}{dx}) - 1 - \frac{dy}{dx} = 0$ so (solving for $\frac{dy}{dx}$):

$$\frac{dy}{dx} = -\frac{\frac{2x}{1+(x^2+y^2)^2} - 1}{\frac{2y}{1+(x^2+y^2)^2 - 1}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{ as well.}$$

(d) $\frac{d}{dx}$ gives $e^y + x\,e^y\frac{dy}{dx} + \frac{1}{1+x} = 0$ so $\frac{dy}{dx} = -\frac{e^y + \frac{1}{1+x}}{x\,e^y} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ too.

22. (a) $\frac{\partial}{\partial x}\arctan\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = -\frac{1}{2}$ and $\frac{\partial}{\partial y}\arctan\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{1}{2}$ ... at (1,1)

so (i) $\nabla f = [-\frac{1}{2}, \frac{1}{2}]$ at (1,1)

and (ii) the tangent lplane to $z = \arctan\frac{y}{x}$ is then $z = \text{Arctan}\frac{1}{1} + \left(-\frac{1}{2}\right)(x-1) + \left(\frac{1}{2}\right)(y-1) = \frac{\pi}{4} - \frac{x-y}{2}$.

(b) $\frac{\partial}{\partial x} \ln(x^2+y^2) = \frac{1}{x^2+y^2}2x = 0$ and $\frac{\partial}{\partial y} \ln(x^2+y^2) = \frac{1}{x^2+y^2}2y = 2$ ... at (0,1)

so (i) $\nabla f = [0,2]$ at (0,1)

and (ii) the tangent plane is $z = \ln 2 + (0)(x-0) + 2(y-1) = \ln 2 + 2y - 2$.

(c) $\frac{\partial}{\partial x}$ f(x,y) = sec x tan x = 0 and $\frac{\partial}{\partial x}$ f(x,y) = sec y tan y = 0 ... at (0,0)

so (i) □ f = [0,0] at (0,0)

and the tangent plane is z = sec 0 tan 0 + 0(x-0) + 0(y-0) = 0 (i.e. the tangent plane is the x-y plane).

(d) $\frac{\partial}{\partial x}$ f(x,y) = 2 and $\frac{\partial}{\partial y}$ f(x,y) = 3 ... at (1,1)

so (i) □ f = [2,3] at (1,1)

and (ii) the tangent plane is z = 2(1)+3(1) + 2(x-1) + 3(y-1) = 2x + 3y

(i.e the tangent plane at any point on the plane z = 2x + 3y is this plane itself.)

23. □ f = [$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$] = [$4x^3y^5$, $5x^4 y^4$] = [4,5] at the point (1,1). The directional derivative (or rate of change) in the direction θ is then 4 cos θ + 5 sin θ .

(a) for θ = 0, get 4 (b) for θ = π, get - 4 (c) for θ = $\frac{\pi}{4}$, get $\frac{9}{\sqrt{2}}$ (d) for θ = $\frac{3\pi}{4}$, get $\frac{1}{\sqrt{2}}$

24. □ f = [f_1(a,b), f_2(a,b)] = [1,$\sqrt{3}$] so the directional derivative in the θ-direction is f_1(a,b) cos θ + f_1(a,b) sin θ = cos θ + $\sqrt{3}$ sin θ = [1,$\sqrt{3}$] • [cos θ,sin θ] ... the DOT product between the vector □ f = [1,$\sqrt{3}$] and the unit vector [cos θ,sin θ]. (Note: we use the notation f_1 to indicate $\frac{\partial f}{\partial x}$, etc.)

(a) This DOT product is 0 if [cos θ,sin θ] is perpendicular to □ f = [1,$\sqrt{3}$].
[1,$\sqrt{3}$] is in the direction $\frac{\pi}{3}$, so θ must be in a direction $\frac{\pi}{3} + \frac{\pi}{2} = \frac{5}{6}\pi$ (+ any integer multiple of π).

(b) The DOT product is as small as possible if [cos θ,sin θ] is opposite to the direction of □ f .
Then θ must be in a direction $\frac{\pi}{3} + \pi = \frac{4}{3}\pi$ (Note: integer multiples of 2π may be added to this angle.)

(c) The DOT product is a large as possible if [cos θ,sin θ] is in the direction of □ f .
Then θ must be in a direction $\frac{\pi}{3}$ (Note: integer multiples of 2π may be added to this angle.)