EXAMPLE PROBLEMS

(done in the text)

- Assuming that the population of a species increases according to $\frac{dN}{dt}$ = k N, determine N(t).

- A spherical mothball evaporates at a rate proportional to its surface area. If it starts with a radius of 2 cm and is reduced to .6 cm after 10 hours, how long will it take to evaporate completely?

- An Egyptian scroll is discovered in which the ratio of ^{14}C to ^{12}C is .6 of the value it would have in similar material today. Estimate the age of the scroll.

- Sketch the DE PORTRAIT for $\frac{dy}{dx} = x^2 + y^2$.

- Water flows into lake Ontario at the rate of A *metres*^{3}*/day* (from rivers and rain, etc. as well as from liquid industrial waste) and this water has an average pollution concentration of B *kg/metres*^{3}. Water (mixed with pollutants) is also withdrawn at the rate of C *metres*^{3}*/day*. Describe, via a differential equation, the amount of pollutants at time t days (after measurements begin).

- Show that the series $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges.

- Determine the Taylor polynomials for f(x) = *ln* x about x = 1.

- Prove that the series $\sum_{n=1}^{\infty} \frac{n!}{x^n}$ diverges for every value of x different from 0.

- Sketch $x = t^2 - 1$, $y = t^3$.

- For the polar curve r = sin (2θ), determine the slope at $\theta = \frac{\pi}{4}$. Also, express as a definite integral the length of the curve from θ = 0 to $\theta = \frac{\pi}{2}$.

- You are standing on the side of a mountain whose elevation is given by $z = 95 - x^2 - y^2 + 2x + 4y$ metres, where x = 0, y = 0 is your location, so z = 95 is your elevation. Sketch the level curves in your neighbourhood and determine in what direction you should climb so as to increase your elevation most rapidly.

- The pressure of a gas P depends upon its volume V and temperature T according to PV = kT where k is a constant. If P = 1, V = 2 and T = 3, how rapidly is the pressure changing when V alone changes?

- The density of ants at a location (x,y) is given by $D(x,y) = K \frac{e^{-x}}{1+y^2}$ *ants per metres*^{2} where x and y measure distance (in *metres*) from the queen ant who is located at (0,0): x is east-west and y is north-south distance. What is the rate of change of ant density at (0,0), in a north-west direction?

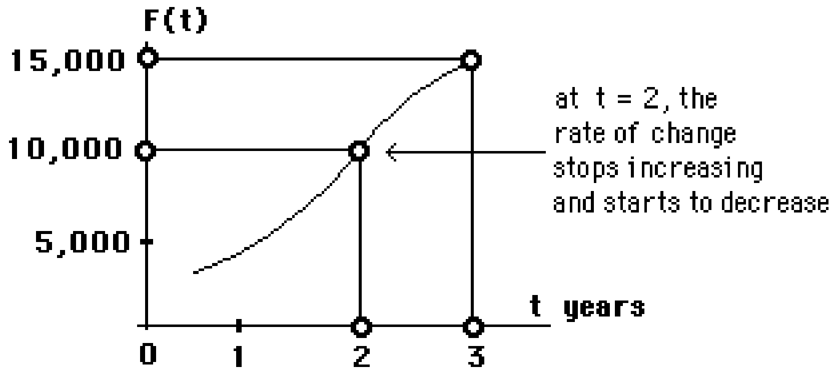
- Examine $z = xy + \frac{1}{x} + \frac{1}{y}$ for a minimum, in x > 0, y > 0.

- Determine the line of "least squared error" to fit the points (.2,.6), (.6,.9), (.9,1.5) and (1.2,1.7).

ASSORTED PROBLEMS

(which you'll be able to solve by the end of this course)

1. When observed over some time interval, it is noted that the population of fish in a lake changes as shown below. Assuming a logistic population growth: $F(t) = \dfrac{K}{1+\left(\dfrac{K-F_0}{F_0}\right)e^{-rt}}$, determine K, the carrying capacity of the lake, and the values of F(0) and r. (You may find it convenient to let $y = \dfrac{F(t)}{K}$.)



2. Solve the following DEs:
   (a) $3x^2y + 2x + x^3\dfrac{dy}{dx} = 0$
   (b) $\dfrac{dy}{dx} + 2xy = x$

3. Suppose that the elevation of a mountain is described by $5x^2 + 4xy + 2y^2 = C$ (where C is the elevation). Determine a DE satisfied by the curve through (2,5) down which you should ski so as to achieve the steepest descent. (i.e determine the DE for orthogonal trajectories for this family of curves.) DO NOT SOLVE. (It's neither *separable* nor *linear*!)

4. Sketch some level curves of the function $f(x,y) = 1 - x^2 + y^2$.

5. Obtain the equation of the tangent plane to the surface $z = x^2+xy$ at the point (1, 2, 3).

6. Determine whether the given series converges or diverges.
   If it converges, determine whether it converges *absolutely* .

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^n - n^\pi}$ (b) $\displaystyle\sum_{i=1}^{\infty} e^i$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,e^n}$ (e) $\displaystyle\sum_{i=1}^{\infty} \frac{i^2}{1+i\sqrt{i}}$ (f) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin n\pi}{\ln n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

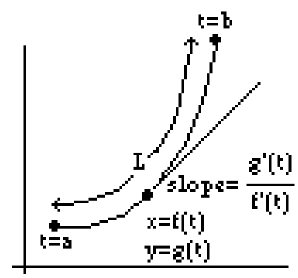
7. Solve each of the following:

(a) $\frac{dy}{dx} = x\, e^y$ (b) $\frac{dy}{dx} = 1 - y^2$

(c) $\frac{dy}{dx} + xy = x$ (d) $\frac{dy}{dx} = 2\,\frac{y}{x}$

8. Suppose that f(t) and g(t) are continuous functions of t for each t in the interval (a,b). The equations x = f(t), y = g(t) give a point (x,y) in the plane and the set of all such points, for $a \le t \le b$, is called a *parametric curve*.

If f(t) and g(t) are differentiable, the slope of the tangent line to this curve at



(x(t),y(t)) is : $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Further, the *length* of the curve, from t = a to t = b, is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For each of the following: (i) *plot* the parametric curve for $a \le t \le b$

(ii) determine $\frac{dy}{dx}$ at $t = t_0$

(iii) calculate the length of the curve from t = a to t = b

(a) $x = 2\cos t$, $y = 2\sin t$, $(a,b) = (0,2\pi)$, $t_0 = \pi/4$

(b) $x = a\cos^3 t$, $y = a\sin^3 t$, $(a,b) = (0,2\pi)$, $t_0 = \pi/4$

(c) $x = 2\cos t$, $y = 3\sin t$, $(a,b) = (0,2\pi)$, $t_0 = \pi/2$ (leave the arc length, L, as an integral!!)

9. Obtain $P_n(x)$, the Taylor polynomial of degree n, for f(x), about x = a, as indicated below:

(a) $f(x) = \sin x$, n = 6, a = 0 (b) $f(x) = \sin x$, n = 6, $a = \frac{\pi}{2}$

(c) $f(x) = e^x$, n = 4, a = 1 (d) $f(x) = \ln(1+x)$, n = 4, a = 0

(e) $f(x) = \tan x$, n = 5, a = 0 (f) $f(x) = \arcsin x$, n = 3, a = 0

(g) $f(x) = \sqrt{1+x}$, n = 3, a = 0 (h) $f(x) = \sinh x$, n = 5, a = 0

(i) $f(x) = \cosh x$, n = 5, a = 0 (j) $f(x) = \frac{1+x}{1-x}$, n = 3, a = 0

Note: In (h) and (i), $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Note: $\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$.

10. For each of the following, determine the Taylor series about x = a and determine its interval of convergence:

(a) $f(x) = \ln(1+x)$, a = 0 (b) $f(x) = \ln x$, a = 1

(c) $f(x) = e^{2x}$, a = 0 (d) $f(x) = \sin x$, $a = \frac{\pi}{4}$

(e) $f(x) = \frac{1}{1-x}$, a = 0 (f) $f(x) = \sinh x$, a = 0

11. Assume that the solution to the differential equation $\frac{dy}{dx} = y$ may be expanded in a power series:

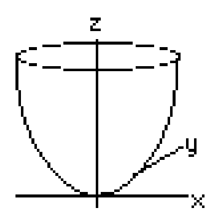
$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_0^{\infty} a_n x^n$. Substitute this series into $\frac{dy}{dx} = y$ and find constants

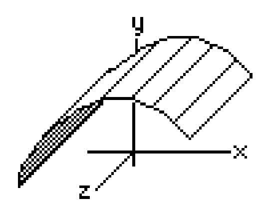
a_0, a_1, a_2, ... , a_n, ... so as to satisfy this differential equation.
(i.e. so that $\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$, for all x.)

12. Use 5 terms of the Maclaurin series (i.e. the Taylor series expanded about x = 0) for e^x to approximate e and show that the error is less than $\frac{1}{5!}\left(1 + \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + ...\right)$ ... and evaluate this error estimate.

13. The well-known physicist, Richard Feynman, writes:
"*Thus the average energy is* $<E> = \frac{h\omega}{2\pi}\,\frac{x + 2x^2 + 3x^3 + ...}{1 + x + x^2 + ...}$ *. Now the two sums which appear here we shall leave for the reader to play with and have some fun with. When we are all finished summing and substituting for x in the sums, we should get - if we make no mistakes in the sum-* $<E> = \frac{\frac{h\omega}{2\pi}}{e^{h\omega/2\pi kT} - 1}$ *. This, then, was the first quantum mechanical formula ever known, or discussed, and it was a beautiful culmination of decades of puzzlement.*" Have some *fun* and obtain the expression for <E>, given that $x = e^{-h\omega/2\pi kT}$. You can easily sum the geometric series $1 + x + x^2 + ...$ but you'll have *fun* with $x + 2x^2 + 3x^3 + ...$ but note that it's almost (but not quite) the series obtained by differentiating $1 + x + x^2 + x^3 + ...$

14. If the graph of $z = f(x^2)$, in the x-z plane, is revolved about the z-axis, the 3-D surface generated is described by $z = f(x^2 + y^2)$. (For example, the parabola $z = x^2$ generates the paraboloid $z = x^2 + y^2$).





Further, the 3-D surface described by y = f(x) (in an x-y-z coordinate system) is a cylinder generated by moving the 2-D curve y = f(x) (in the x-y plane) parallel to the z-axis.

Use this to sketch the graph of the following functions. Identify the surface as a paraboloid, hyperboloid, cylinder, plane or cone.

(a) $z = 1 - x^2 - y^2$ (b) $z = \sqrt{x^2 + y^2}$

(c) $z = x$ (d) $y = x^2$

(e) $z^2 = 1 + x^2 + y^2$ (f) $z^2 = 1 + x^2 - y^2$

15. Sketch some level surfaces, f(x,y) = constant, for each of the following:

(a) $f(x,y) = x - y$ (b) $f(x,y) = x^2 + 2y^2$

(c) $f(x,y) = x\,e^{-y}$ (d) $f(x,y) = \frac{1}{x^2 - y^2}$

16. Find the equation of the tangent plane to the graph of the given function at the point indicated:

(a) $z = \arctan\left(\frac{y}{x}\right)$ at (-1,1) (b) $z = \frac{1}{\sqrt{x^2 + y^2}}$ at (-3,4)

(c) $z = \frac{2\,x\,y}{x^2 + y^2}$ at (0,2) (d) $z = \ln(1 + e^{xy})$ at (0,1)

17. If u(x,t) represents the temperature at the position x at time t in an insulated rod, then u satisfies the so-called Heat Equation: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ where the constant k is called the diffusivity (related to the thermal conductivity).

Show that each of the following satisfies the Heat Equation for *some* value of k and determine this k-value.

(a) $u(x,t) = e^{-2t} \sin x$ (b) $u(x,t) = \frac{e^{-x^2/4t}}{\sqrt{t}}$

18. If u(x,y,t) represents the elevation of a vibrating elastic membrane above the x-y plane (think of a drum head) at the position (x,y) at time t, then u satisfies the so-called Wave Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ where c is the speed of propagation of waves on the membrane. Show that each of the following satisfies the Wave Equation for *some* c>0 and determine this c-value:

(a) $u(x,y,t) = \sin x \sin 2y \cos 3t$ (b) $u(x,y,t) = e^{3x + 4y + 5t}$

19. Use the Chain Rule to determine the indicated derivative:

(a) $\frac{dz}{dt}$ when t = 0 if $z = x^2 + y^2$ and $x = e^t$, $y = \sin t$

(b) $\frac{dz}{dt}$ when $t = \pi$ if $z = x^2 + y^2$ and $x = \cos t$, $y = \sin t$

(c) $\frac{dw}{dz}$ when z = 1 if $w = u^2 + u\,v$ and $u = \ln z$, $v = z^2$

(d) $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$ if $y = u \operatorname{Arctan} v$ and $u = \sin x$, $v = \tan x$

(e) $\frac{dw}{dt}$ when $t = \frac{\pi}{2}$ if $w = x^2 + y^2 + z^2$ and $x = \sin t$, $y = \cos t$, $z = t$

20. The elevation of a mountain is given by $z = 1000 - x^2 - 2\,y^2$ where z (metres) is the elevation and x and y (each measured in metres) are the horizontal distances from the centre of the mountain. If a person is located at $x = 3 + 2t$, $y = t^2$, at time t (in seconds), then find the rate at which the person's elevation is changing (in metres per second) when t = 2 seconds.

21. For each of the following determine $\frac{dy}{dx}$: (i) by implicit differentiation, and (ii) by using $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$:

(a) $f(x,y) = x^2 + y^2 - a^2 = 0$ (b) $f(x,y) = x^2\,y + \sin(x\,y) - x = 0$

(c) $f(x,y) = \arctan(x^2 + y^2) - x - y = 0$ (c) $f(x,y) = x e^y + \ln(1 + x) - 1 = 0$

22. The gradient of a function f(x,y) is a *vector* with x- and y-components $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ respectively.

i.e. $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$. For each of the following functions determine :

(i) the gradient vector at the given point and

(ii) the equation of the tangent plane to the surface z = f(x,y) at the given point:

(a) $f(x,y) = \arctan \frac{y}{x}$ at (1,1) (b) $f(x,y) = \ln(x^2 + y^2)$ at (0,1)

(c) $f(x,y) = \sec x + \sec y$ at (0,0) (d) $f(x,y) = 2x + 3y$ at (1,1)

23. The rate of change of f(x,y) in the direction θ (the directional derivative) is $\frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta$.

Compute the rate of change of $f(x,y) = x^4 y^5$ at (1,1) in the given direction:

(a) $\theta = 0$ (b) $\theta = \pi$ (c) $\theta = \frac{\pi}{4}$ (d) $\theta = \frac{3\pi}{4}$

24. If $\frac{\partial f}{\partial x}(a,b) = 1$ and $\frac{\partial f}{\partial y}(a,b) = \sqrt{3}$, in what direction, θ, should a directional derivative be computed, at (a,b), in order that it is:

(a) 0 (b) as small as possible (c) as large as possible (Note: choose θ in $(0,2\pi)$).

LECTURE 1

INTRODUCTION TO DIFFERENTIAL EQUATIONS

POPULATION GROWTH:

PS:

P: How would you describe, in mathematical terms, the growth of a population?

S: Huh?

P: If I said the population of a city or a country or the world was growing at 2% per year, how would you put this into mathematical terms?

S: I'd say ... uh ... I don't understand the question.

P: Okay, suppose the population was N people after a time t years ... "t" could be the time, in years, since you started to measure the population; maybe t = 0 corresponds to the year 1925 so t = 5 corresponds to 1930 and so on. Every year the population grows by 2% so that, if there are N people at the time t, there will be 1.02 N one year later and the increase in population will be .02N people per year. See? *People per year*. This .02N is a rate of change of population with respect to time and it's proportional to the current population so we could write: $\boxed{\frac{dN}{dt} = k\,N}$ where "k" is some constant of proportionality (like .02, for example). Now, the big question: what must be the function N(t) so that it satisfies $\frac{dN}{dt} = kN$? What do you think?

S: I haven't the foggiest.

The equation $\frac{dN}{dt} = k\,N$ is called a *differential equation*. Any equation involving some unknown function and its derivatives is called a differential equation (or DE for short). In our case the unknown function is N(t). (We assume that the constant "k" is known.)

Example DEs: (in what follows, "k" and "K" and "C" are constants)

- $\frac{ds}{dt} = .98\,t$ is a DE which describes the distance fallen by an object (namely s(t) metres) in a time t seconds.
- $\frac{dy}{dx} = k\,\frac{y}{x}$ is a DE which describes the relationship between the size of a persons eye (namely y) in relation to the size of the head (namely x).
- $\frac{dT}{dt} = -\,k\,(T - K)$ is a DE which describes the temperature T of some object which is cooling in air which is at a temperature K.
- $\frac{dN}{dt} = -\,k\,N\,(N - K)$ is a DE (the so-called "Logistic Equation") which is an alternate description of the growth of populations. Here N(t) is the population at time t and the constant K is called the "carrying capacity".

It is often the case that the mathematical description of some process (like population growth) is most easily obtained as a DE. The problem is: given a DE involving some unknown function, what is this function which "satisfies" the DE?" or, to put it differently, "what are the *solutions* to the given DE?"

- $s(t) = .49\,t^2$ is a solution to the DE: $\frac{ds}{dt} = .98\,t$ ... and so is $s(t) = .49\,t^2 + 30$ or $s(t) = .49\,t^2 - \pi$. (You just substitute s(t) into the DE to see if $\frac{ds}{dt} = .98\,t$)
- $y = x^k$ is a solution to the DE: $\frac{dy}{dx} = k\,\frac{y}{x}$ since $\frac{dy}{dx} = k\,x^{k-1}$ and $k\,\frac{y}{x} = k\,\frac{x^k}{x} = k\,x^{k-1}$ as well. Note, however, that $y = 10x^k$ is also a solution as are $y = \pi x^k$ and $y = -\,47\,x^k$. In fact, $y = C\,x^k$ satisfies the DE for any choice of constant "C".
- $T(t) = K + e^{-kt}$ is a solution of $\frac{dT}{dt} = -\,k\,(T - K)$, and so is $T(t) = K - e^{-kt}$ and $T(t) = K - 39\,e^{-kt}$ and, in fact, $T(t) = K + C\,e^{-kt}$ satisfies the DE $\frac{dT}{dt} = -\,k\,(T - K)$ for any choice of constant C.

- $N(t) = \frac{K}{1 + Ce^{-kKt}}$ is a solution of the logistic equation $\frac{dN}{dt} = - k N (N - K)$ for any choice of constant C.

We note several things:

(1) There are many solutions to a given DE.
(2) The solutions all have a constant which can have any value. Every value assigned to this constant (called "C" above) gives a solution.
(3) Given a DE and a solution, it is a simple matter to check that it is a solution: just substitute back into the DE!

Now, the big problem: HOW DO WE FIND THE SOLUTIONS?

PS:

S: Are DEs really that important? I mean, do they *really* pop up in *real* problems or are they just toys for the mathematicians to play with?

P: It's a curious thing, but many *real* problems are more easily stated in terms of rates of change ... or derivatives, and that gives rise to DEs. In trying to find some quantity (like the population N(t) or the distance s(t)) we often know something about its rate of change, like "the rate of change of population is 2% per year". Now it's up to us to find the quantity itself ... from the DE. For that reason we'll study a few simple types of DEs and their method of solution.

The World's Simplest DE:

If we're lucky enough to have a DE such as $\frac{ds}{dt} = .98\,t$, we just integrate both sides with respect to t. In this case, s(t) is simply an antiderivative or indefinite integral of .98 t: $s(t) = \int .98\,t\,dt = .98\frac{t^2}{2} + C$ where we've added a constant of integration, C. In general these "World's Simplest DEs" have the form: $\frac{dy}{dx} = f(x)$ and the solutions are just $y = \int f(x)\,dx$ where we'll add a constant C after performing the integration ... and the constant can have any value, so we have a whole family of solutions, one for each choice of the constant.

The DE $\frac{dy}{dx} = f(x)$ has solutions $y = \int f(x)\,dx + C$

Although not necessary, we've indicated the constant of integration "C" even before we've completed the integration ... just to remind ourselves that it must be there!

Example: A body falls from rest under the influence of a constant gravitational acceleration of .98 metres/second^2. Determine its velocity and position at any time t seconds.

Solution: Let v(t) be the downward velocity at time t, then $\frac{dv}{dt} = .98$ (Note that v is measured in m/s so $\frac{dv}{dt}$ is measured in m/s^2 and we're happy because that's what the gravitational acceleration is measured in!) Now we integrate to get $v(t) = \int .98\,dt = .98\,t + C$. We are given that, initially, at t = 0, the body is at rest meaning that v(0) = 0. This gives us the value of C: we substitute v = 0 and t = 0 and get 0 = 0 + C so C = 0, hence v(t) = .98 t m/s. Now let s(t) be the distance fallen (in metres) so that $\frac{ds}{dt} = v = .98\,t$, hence (integrating) we get $s(t) = .49\,t^2 + C_1$ (where we've called the constant of integration C_1 so as not to confuse it with the previous constant, C). At t = 0 the body hasn't fallen at all hence s(0) = 0 so we substitute s = 0, t = 0 and find that $0 = 0 + C_1$ so $C_1 = 0$ and the distance fallen is now: $s(t) = .49\,t^2$ metres.

Note: Had the gravitational acceleration been given as g m/s^2 (rather than .98 m/s^2) we'd get $\frac{dv}{dt} = g$ and $v = g\,t$ and $s = \frac{1}{2}\,g\,t^2$ ... and this would be good on any planet (where g might be different than on this planet).

SEPARABLE DIFFERENTIAL EQUATIONS:

The world's second simplest DE has the form: $\frac{dy}{dx} = f(x)\, g(y)$ where the right-side can be factored into the product of a function only of y and a function only of x. These, too, are easy to solve:

Example: Solve $\frac{dy}{dx} = k \frac{y}{x}$.

Solution: We "separate the variables", collecting on the left-side all functions of y together with the "dy" and leaving the rest on the right-side together with the "dx", and rewrite the DE in the form: $\frac{dy}{y} = k \frac{dx}{x}$. Now we integrate each side: $\int \frac{dy}{y} = \int k \frac{dx}{x}$ hence $\ln |y| = k \ln |x| + C$. In many cases we can't find y explicitly, but here we can ... and we do. Exponentiating each side we get $|y| = e^{k \ln |x| + C} = e^C e^{\ln |x|^k} = C_1 |x|^k$. This gives our family of solutions: $| y | = C_1 | x |^k$ where, since the constant of integration, C, was arbitrary, then so is e^C which we've rewritten as C_1. Note, however, that $C_1 = e^C$ implies that C_1 is positive (since e^C is always positive). Note, too, that if $| y | = C_1 | x |^k$, then $y = \pm C_1 | x |^k$ (a property of the absolute value!). Hence we can replace | y | by y if we admit positive and negative constants C_1. We do this, and now have our solutions: $y = C_2 | x |^k$ where, of course, C_2 can be any constant, either positive or negative. If we knew that x was positive (as would be the case if x measured the size of a person's head!), then | x | = x (a property of the absolute value!) and we could write our solutions as: $y = A x^k$ (where using "A" as the name of our "arbitrary constant" looks nicer than using C_2).

PS:

S: Whoa! That's real confusing, isn't it? I mean, am I supposed to be able to do all that ... like talk about why we can drop the absolute value signs and all that stuff?

P: In many cases you'll know beforehand whether the quantities are positive, then you can just write $\int \frac{dy}{y} = \ln y$ and forget about using $\ln | y |$. Sometimes, however, the quantities can take on negative values (perhaps x is a temperature and it's below zero) so you have to keep the absolute values in order not to eliminate any solutions. For example, had we written the solutions as $y = A x^k$ and x were negative and $k = \frac{1}{2}$ then we'd be in big trouble wouldn't we?

S: We would?

P: Sure, because you'd write $y = A \sqrt{x}$ as the solution and x is negative and that's bad news, right? In fact, you'd *really* want to write your solution as $y = A \sqrt{|x|}$ so y would be defined even for negative x-values.

S: Okay, but tell me ... how can you just "collect on the left-side all functions of y together with the dy"? Is that legal? I mean, you're breaking $\frac{dy}{dx}$ into two pieces, the "dy" and the "dx" and stuff like that. Can you do that?

P: That's the technique for solving these types of "separable" DEs (i.e DEs where you can "separate the variables"). I can do it differently if you'd like. Suppose I write $\frac{dy}{dx} = k \frac{y}{x}$ as $\frac{1}{y}\frac{dy}{dx} = k \frac{1}{x}$ then I recognize the left-side as the derivative of $\ln | y |$ with respect to x. I can then rewrite the DE as: $\frac{d}{dx} \ln | y | = k \frac{1}{x}$. Now I say: "If $\frac{d}{dx} \ln |y|$ is $k \frac{1}{x}$, then $\ln | y |$ must be an antiderivative, or indefinite integral, of $k \frac{1}{x}$, so I'd put $\ln | y | = \int k \frac{1}{x} dx = k \ln | x | + C$ and I'd get the same answer, right?

S: I'm sorry I asked. I think I'll stick with "collecting the y's with the dy" .

The separable DE $\frac{dy}{dx} = f(x)\, g(y)$ has solutions satisfying $\int \frac{dy}{g(y)} = \int f(x)\, dx + C$

Example: Assuming that the population of a species increases according to $\frac{dN}{dt} = k\,N$, determine N(t).

Solution: This is a separable DE so we "separate the variables", writing: $\frac{dN}{N} = k\,dt$ then integrate each side to get $\int\frac{dN}{N} = \int k\,dt$ or $ln\ N = k\,t + C$ (where we know that N > 0 so we omit the absolute value sign).

Exponentiating each side gives: $N = e^{kt+C} = e^C e^{kt} = A\,e^{kt}$ (where we've replaced the "arbitrary constant" e^C by A). We conclude that the population grows exponentially. In fact, if N = 10,000 at t = 0, then $10{,}000 = A\,e^0 = A$ and we've identified the constant A for this particular population, so we'd write $N(t) = 10{,}000\,e^{kt}$.

S: Yeah, you've found A but you haven't found k.

P: The DE had a constant k and the solution adds another constant C (or A) so we'd need two pieces of information about our population in order to solve for these two constants. One piece was given, namely the initial population of 10,000 and that enabled us to determine A. What other piece of info would you like to provide?

S: Huh?

P: Suppose we actually count the population after, say, 5 years, and find it to be 13,000. Then we'd have N(0) = 10,000 (the initial population) and N(5) = 13,000 and that'd give us two equations to solve for the two constants A and k, namely: $10{,}000 = A\,e^0$ and $13{,}000 = A\,e^{5k}$. The first gives A = 10,000 and the second gives $e^{5k} = \frac{13000}{10000} = 1.3$ so $5k = ln\,(1.3)$ so $k = \frac{ln\ 1.3}{5}$ and $N(t) = A\,e^{kt} = 10{,}000\,e^{(t/5)ln\ 1.3} = 10{,}000\,e^{ln(1.3)^{t/5}} = 10{,}000\,(1.3)^{t/5}$ and there you have it!

S: Have what? What'd you do up there in the exponent?

P: I just used properties of logarithms and exponents: $q\,ln\ p = ln\ p^q$ is one, and $e^{ln\ p} = p$ is another, so whenever I see $e^{a\ ln\ b}$ I write the exponent as $ln\ b^a$ (using the first property) and then $e^{ln\ b^a} = b^a$ (using the second property). But watch this: we can find k from the equation $e^{5k} = 1.3$ (as we did above), but what we really want is e^{kt} (not k itself). From $e^{5k} = 1.3$ we get $e^k = (1.3)^{1/5}$ so $e^{kt} = (1.3)^{t/5}$. See how simple?

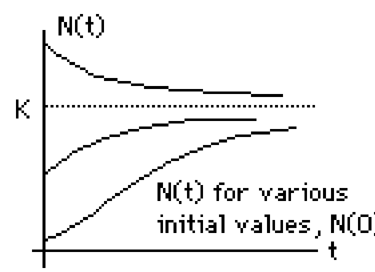
Example: The population of a certain species of clam is estimated at 10,000,000 then, a year later, at 17,000,000. What would the expected population be after 10 years?

Solution: We'll call the population N(t) after a time of t years, and we'll assume it satisfies the differential equation $\frac{dN}{dt} = k\,N$ where k is some as-yet-unknown constant. The solution (as obtained earlier) is $N(t) = A\,e^{kt}$.

We have two constants and two pieces of information: $N(0) = 10{,}000{,}000 = A\,e^0$ and $N(1) = 17{,}000{,}000 = A\,e^k$. We solve for A = 10,000,000 and $e^k = 1.7$ so that $N(t) = 10{,}000{,}000\,(e^k)^t = 10{,}000{,}000\,(1.7)^t$. When t = 10 we get a population of $N(10) = 10{,}000{,}000\,(1.7)^{10}$ which is about 2×10^{19} clams.

S: That's a lot of clams!

P: That's exponential growth for you. Of course, maybe the population doesn't satisfy $\frac{dN}{dt} = k\,N$ (which describes exponential growth). In fact, we'd expect that after a while there would be too many clams and too little food for them to eat and they'd die off like ... uh, flies. The DE doesn't take that into account; it just assumes a rate of growth that keeps on going and going. Maybe what we need is a DE whose solutions grow, then level off at some value. That's maybe what one would expect, right? See the diagram? N(t) increases rapidly at first, then begins to level off and approaches some value: K clams. Further, if N is too large, say N > K, then N should decrease (because the clams would die off due to lack of food). There is some population, namely N = K, where the population remains steady. Too many clams and they die off. Too few and they increase in number. What kind of DE would have such a solution?



S: I haven't the foggiest.

We start with $\frac{dN}{dt}$ = *something* and this *something* should be positive when N < K (because the solution is increasing) and negative when N > K. The simplest such DE would be $\frac{dN}{dt} = K - N$ or, to make it more general we might consider $\frac{dN}{dt}$ = k (K - N) where k > 0 is some constant. To solve this DE we would separate the variables, writing $\frac{dN}{K - N}$ = k dt, then integrate to get -*ln* | K - N | = kt + C so |K - N| = e^{-kt} e^{-C} so K - N = ± e^{-C} e^{-kt} = A e^{-kt} hence we have our solution: N(t) = K - A e^{-kt} and we notice that N(t) –> K as t –> ∞ as we expected. Further, if the initial population is given as N(0) at t = 0, we'd want N(0) = K - A so A = K - N(0) and, finally,

$$\boxed{N(t) = K - (K - N(0))\, e^{-kt}}$$

. Although this has some of the properties we want, it also suggests that N(t) –> K as t –> ∞ even if there are initially no clams! (i.e N(0) = 0). That's not good. If there are no clams at t = 0 there will *always* be no clams.

What we want, then, is a DE where N = 0 is also a solution. One of the simplest is $\frac{dN}{dt}$ = - k N (N - K) where k and K are constants.

P: See? If N < K then $\frac{dN}{dt}$ > 0 and when N > K then $\frac{dN}{dt}$ < 0 and N(t) = 0 is a solution as well. Do you recognize this DE?

S: Nope.

P: It's the logistic equation. Note how nice it is. When the population is small, then N - K is approximately K (neglecting "N" compared to "K") so the DE looks like $\frac{dN}{dt}$ = - k N (- K) = kK N which, as we've seen, has exponentially growing solutions and that's exactly what we want. After all, we'd expect populations to grow exponentially until there are too many individuals for the environment to support ... limited food supply, etc. Let's solve it ... which shouldn't be too difficult since it's separable.

We rewrite the DE as: $\frac{dN}{N(N - K)}$ = - k dt and integrate to get $\int \frac{dN}{N\,(N - K)}$ = - kt + C. The integral on the left-side isn't one we've met before, but we can write $\frac{1}{N\,(N - K)} = \frac{1}{K}\left(\frac{1}{N - K} - \frac{1}{N}\right)$ and integrate these simpler expressions. We'd get: $\frac{1}{K}\int\left(\frac{1}{N - K} - \frac{1}{N}\right) dN = \frac{1}{K}\left(ln\,|N - K| - ln\ N\right) = \frac{1}{K}\ ln \frac{|N - K|}{N}$ where we've used log A - log B = log $\frac{A}{B}$. Note that we kept the absolute value sign around N - K because N could be greater or less than K. Okay, now we have $\frac{1}{K}\ ln \frac{|N - K|}{N}$ = - kt + C so $ln \frac{|N - K|}{N}$ = - kKt + C_1 (where we write the arbitrary constant KC as C_1) and now we exponentiate to get $\frac{|N - K|}{N}$ = e^{C_1} e^{-kKt} = A e^{-kKt} (where, again, we've relabelled our arbitrary constant). Now A = e^{C_1} is always positive but we can eliminate the absolute value sign to get

$\frac{N - K}{N}$ = ± A e^{kKt} and then absorb the ± into the constant A, letting it be either positive or negative, and then we solve for N(t) via: 1 - $\frac{K}{N}$ = A e^{-kKt} so $\frac{K}{N}$ = 1 - A e^{-kKt} so, finally, N(t) = $\frac{K}{1 - Ae^{-kKt}}$ and we can even absorb another (-) into A and write:

$$\boxed{N(t) = \frac{K}{1 + A\,e^{-kKt}}}$$

as the solution to the logistic equation, where k, K and A are constants. We note that $\lim_{t \to \infty}$ N(t) = K (since e^{-kKt} –>0) so the eventual population does indeed approach this constant, limiting value.

S: You're doing a lot of this "absorbing" with your constants. Is that legal?

P: Well, I could write ±A = B (i.e. use a different symbol) then later get $\frac{1}{1-Be^{-kKt}} = \frac{1}{1+Ce^{-kKt}}$ using C to represent -B. See? Some people start off with their first constant labelled C_1, then introduce C_2, C_3, etc. as needed. Come to think of it, you should do that, just so you don't get confused.

S: Another thing. How'd you integrate that $\frac{1}{N(N-K)}$? You didn't teach me that ... did you?

P: You're right, I didn't. But let's do it in the next lecture ... you need a rest.

<u>LECTURE 2</u>

MORE ON DIFFERENTIAL EQUATIONS

a Little Partial Fractions:

The expression $\frac{Ax+B}{(x+a)(x+b)}$ can always be written in the form $\frac{\textit{something}}{x+a} + \frac{\textit{something}}{x+b}$ where the two *something*s are constants. To determine what they are, just bring the right-side to a common denominator, add, and make sure it agrees with $\frac{Ax+B}{(x+a)(x+b)}$.

Example: Find constants P and Q so that $\frac{2x-3}{(x+1)(x-2)} = \frac{P}{x+1} + \frac{Q}{x-2}$.

Solution: First we bring the right-side to a common denominator and add: $\frac{P}{x+1} + \frac{Q}{x-2} = \frac{P(x-2) + Q(x+1)}{(x+1)(x-2)}$ and note that the common denominator is just the original denominator. (It always happens that way!) Then we choose P and Q so the numerators are also equal: 2x - 3 = (P+Q) x + Q - 2P which requires 2 = P+Q and -3 = Q-2P. We solve these <u>two</u> equations in <u>two</u> unknowns for P and Q to get: P = 5/3 and Q = 1/3. (We also check that P+Q = 2 and Q-2P = -3 so we know we have the solution!) Hence we have: $\boxed{\frac{2x-3}{(x+1)(x-2)} = \frac{5}{3}\frac{1}{x+1} + \frac{1}{3}\frac{Q}{x-2}}$ so, for example, we can now easily integrate $\int \frac{2x-3}{(x+1)(x-2)}\,dx = \frac{5}{3}\,ln\,|x+1| + \frac{1}{3}\,ln\,|x-2|$.

S: You forgot the "+C". Anyway, is that what you call "partial fractions"?

P: Any rational function $\frac{p(x)}{q(x)}$ (a ratio of polynomials p and q) where the degree of p(x) is less than the degree of the polynomial q(x) can be written as a sum of fractions, each fraction having as its denominator a factor of q(x). That makes it much easier to integrate, right? And when you bring the sum to a common denominator and add, the common denominator is just q(x) so you just have to match up the numerator so it's p(x). It's fun. Would you like to see more of this partial fractions stuff?

S: Not unless it'll be on the final exam.

Example: Solve the DE: $\frac{dy}{dx} = \frac{y^2-1}{x}$.

Solution: We separate and write $\frac{dy}{y^2-1} = \frac{dx}{x}$ then integrate each side to get $\int \frac{1}{y^2-1}\,dy = ln\,|\,x\,| + C_1$. To integrate the left-side, we write $\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$ (that's the "partial fraction expansion" of $\frac{1}{y^2-1}$)

and add the fractions on the right-side to get $\frac{A(y+1) + B(y-1)}{(y-1)(y+1)} = \frac{(A+B)y + (A-b)}{y^2-1}$ and match this with $\frac{1}{y^2-1}$ so we'd need A+B = 0 and A-B = 1 and we'd solve for $A = \frac{1}{2}$ and B = - $\frac{1}{2}$ (then we'd check that A+B=0 and A-B=1) then we'd integrate: $\int \frac{dy}{y^2-1} = \int \left(\frac{1}{2}\frac{1}{y-1} - \frac{1}{2}\frac{1}{y+1}\right) dy = \frac{1}{2}\, ln\,|y-1| - \frac{1}{2}$ $ln\,|y+1|$ so that $\int \frac{1}{y^2-1}\, dy = ln\,|\,x\,| + C_1$ becomes

$\frac{1}{2}\, ln\,|y-1| - \frac{1}{2}\, ln\,|y+1| = ln\,|x| + C_1$ which can also be written $ln \left|\frac{y-1}{y+1}\right| = ln\, x^2 + C$ (where $C = 2C_1$ and we've written $2ln\,|x| = ln\,|x|^2 = ln\, x^2$ because clearly x^2 is positive and we can drop the absolute value sign). Finally we exponentiate each side and get: $\frac{y-1}{y+1} = \pm e^C e^{ln\, x^2} = K x^2$ and we solve for $y = \frac{1 + K x^2}{1 - K x^2}$ where K is any constant.

S: Am I supposed to be able to do that?
P: Yes.
S: Mamma mia!

Example: A spherical mothball evaporates at a rate proportional to its surface area. If it starts with a radius of 2 cm and is reduced to .6 cm after 10 hours, how long will it take to evaporate completely?

Solution: We have assumed that $\frac{dV}{dt} = k A$ (since the rate of evaporation, $\frac{dV}{dt}$, is proportional to the surface area A). But the volume of the spherical mothball is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$ where r is its radius, so we can write: $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k (4\pi r^2)$ or $4\pi r^2 \frac{dr}{dt} = k (4\pi r^2)$ so that $\frac{dr}{dt} = k$, a constant (which is obviously negative if r is to decrease!). We then solve this world's simplest DE: $\frac{dr}{dt} = k$ to get r(t) = kt + C. We have two constants k and C and need two pieces of information. One is that r(0) = 2 cm; that gives the equation: 2 = 0 + C. The second piece of information is that r(10) = .6 and that gives the second equation: .6 = 10k + C. We solve <u>two</u> equations in <u>two</u> unknowns to get C = 2 and k = - .14 (which, as expected, <u>is</u> negative). Hence r(t) = - .14 t + 2 (and we check to confirm that r(0) = 2 and r(10) = .6). The mothball evaporates completely when r = - .14 t + 2 = 0 and that gives $t \approx 14.3$ hours.

Example: The population of a certain species of clam is estimated at 10,000,000 then, a year later, at 17,000,000, then a year later at 21,000,000. What would the expected population of clams be after 10 years? Assume a logistic growth pattern: $\frac{dN}{dt} = -k N (N - K)$.

Solution: The solution to the logistic equation is: $N(t) = \frac{K}{1 + A e^{-kKt}}$ (as we've already shown) . There are three constants A, k and K and, fortunately, three pieces of information given. We have N(0) = 10,000,000 = $\frac{K}{1+A}$ and N(1) = 17,000,000 = $\frac{K}{1+Ae^{-kK}}$ and N(2) = 21,000,000 = $\frac{K}{1+A\,e^{-2kK}}$ and from these three equations we have to determine the three constants A, k and K in order to compute $N(10) = \frac{K}{1+A\,e^{-10kK}}$. Before proceeding, let's agree to measure clams in millions, then the three equations can be written more simply: $\frac{K}{1+A} = 10$ and $\frac{K}{1+Ae^{-kK}} = 17$ and $\frac{K}{1+A\,e^{-2kK}} = 21$. In fact, to make life even simpler, let's give e^{-kK} a simpler name: let $x = e^{-kK}$ so now our equations read: $\frac{K}{1+A} = 10$ $\frac{K}{1+Ax} = 17$ $\frac{K}{1+Ax^2} = 21$

or perhaps: (1) $1+A = \frac{K}{10}$ (2) $1+Ax = \frac{K}{17}$ (3) $1+Ax^2 = \frac{K}{21}$.

Then (1) - (2) gives ...... (4) $A\,(1-x) = \frac{7}{(10)(17)}\,K$ and

(2) - (3) gives .... (5) $Ax\,(1-x) = \frac{4}{(17)(21)}\,K$

and (5)/(4) eliminates both A and K and leaves a single equation for $x = \frac{4\,(10)(17)}{7\,(17)(21)} = \frac{40}{147}$. Substitute into, say (1) and (2) and get two equations for A and K which, when solved, gives: $A = \frac{1029}{790}$ and $K = \frac{1819}{79}$. We then have $N(10) = \frac{K}{1+A\,e^{-10kK}} = \frac{K}{1+Ax^{10}} \approx 23.0$ which means there should be about 23,000,000 clams after 10 years.

S: Wow! That's a lot of work, isn't it?

P: And a lot of clams. But look at the number of clams: about 23,000,000. Now look at the value of $K = \frac{1819}{79} = 23.025$ million. See? K gives the eventual population, so after 10 years the clam population has pretty well stabilized at its so-called "carrying capacity" of 23 million. Note that this is a far cry from the previous estimate, when we used the DE $\frac{dN}{dt} = k\,N$. That DE gave exponential growth and some 2^{19} clams in 10 years, enough to cover the globe and ..

S: Should I know anything about clams?

P: You should be able to reproduce what I've done ... given enough time. You should certainly be able to *understand* what I've done.

S: I'm in big trouble.

EXPONENTIAL DECAY:

The equation $\frac{dN}{dt} = k\,N$ for population growth may be written $\frac{dN}{dt} = aN - bN$ where aN is the rate of increase in population due to births (and is assumed proportional to the current population) and bN is the rate of decrease due to deaths, so $k = a - b$. If $a > b$ then $k > 0$ and the population increases, but if $a < b$ (meaning the death rate exceeds the birth rate) then $k < 0$ and the solution $N(t) = A\,e^{kt}$ decreases to a limiting value of zero. This is exponential decay.

Radioactive substances also "decay", emitting atomic particles and changing into other elements. Radium, for example, decays into lead and carbon 14 decays into something (which is no longer carbon 14!). In each case the DE which governs the decay is $\frac{dM}{dt} = -\,k\,M$ where M(t) is the amount of substance (radium or carbon 14, etc.) left at time t and we've actually put a (-) out front so we could recognize decreasing solutions, hence "decay". The solution is obtained by "separating the variables": $\frac{dM}{M} = -\,k\,dt$ so $\int\frac{dM}{M} = -\,k\,t + C$ so $\ln M = -kt + C$ so $M = e^C\,e^{-kt}$ which we write as: $\boxed{M(t) = M(0)\,e^{-kt}}$ where M(0) is the initial amount (at time t = 0).

Example: Carbon 14 (denoted by ^{14}C) decays radioactively, any original amount reducing to half after about 5600 years (called the "half-life"). For a plant, it is assumed that the ratio of ^{14}C to ^{12}C is constant* while the plant is living, but that ^{14}C begins to decay as soon as the plant dies (so the ratio decreases). An Egyptian scroll is discovered in which the ratio of ^{14}C to ^{12}C is .6 of the value it would have in similar material today. Estimate the age of the scroll.

* An American scientist, W.F. Libby, won the 1960 Nobel prize for his discovery of carbon dating. Cosmic radiation converts nitrogen in the atmosphere to carbon 14 , plant and animal tissue absorb this radioactive carbon (maintaining a fairly constant ratio of carbon 14 to normal carbon 12), then the tissue dies and the carbon 14 begins its radioactive decay.

Solution: The DE governing the radioactive decay of ^{14}C is $\frac{dM}{dt} = -k M$ with solutions $M(t) = M(0) e^{-kt}$ where the arbitrary constant is M(0) (the initial amount of ^{14}C, measured in kg or any convenient mass unit) and t is measured in years from the time that the radioactive decay began. When t = 5600 the amount of ^{14}C is $\frac{M(0)}{2}$ so we have $\frac{M(0)}{2} = M(0) e^{-5600k}$ and we can solve for $e^{-k} = \left(\frac{1}{2}\right)^{1/5600}$ and the amount of ^{14}C at any time t can now be written: $M(t) = M(0)\left(\frac{1}{2}\right)^{t/5600}$ (and we check that we do indeed get $\frac{M(0)}{2}$ when t = 5600). After t years the amount of ^{14}C in the scroll is only .6 of what it would have been at t = 0 so we write $.6\, M(0) = M(0)\left(\frac{1}{2}\right)^{t/5600}$ and determine t, the age of the scroll. Cancelling M(0) and taking the *ln* of each side and solving gives: $t = -5600 \frac{\ln .6}{\ln 2} \approx 4100$ years. In other words it takes 4100 years to reduce the ratio to .6 and this seems reasonable because, in 5600 years, the ratio would be reduced to 1/2.

Example: Show that the amount of a radioactive substance is $M(t) = M(0)\left(\frac{1}{2}\right)^{t/T}$ where T is the half-life.

Solution: The solutions to $\frac{dM}{dt} = -k M$ are $M(t) = M(0) e^{-kt}$ and $M(T) = \frac{M(0)}{2} = M(0) e^{-kT}$ gives $e^{-k} = \left(\frac{1}{2}\right)^{1/T}$ so that $M(t) = M(0)\left(\frac{1}{2}\right)^{t/T}$.

P: Are you paying attention? See? The number $\frac{1}{2}$ is here more pertinent than the number "e".

S: What? I wasn't listening.

P: Remember when I said that "e" occurs in *real* problems in the form e^{ax}, not just e^x, and that e^{ax} can always be written 10^{bx} or 2^{cx} or π^{dx} or whatever. That is, you can always find numbers "b" or "c" or "d" so the base is almost anything you want ... and that makes "e" seem less significant, right?

S: If you say so.

Newton's Law of Cooling

If it is assumed that the temperature of a hot object cools at a rate proportional to how much hotter it is than its surroundings, then we get Newton's Law of Cooling: $\frac{dT}{dt} = -k(T - K)$ where K is the temperature of the surroundings (called the *ambient temperature*). This is a separable DE and can be easily solved: $\frac{dT}{T - K} = -k\, dt$ so $\int \frac{dT}{T - K} = -k t + C$ so $\ln |T - K| = -k t + C$ so $|T - K| = e^{-kt+C} = e^C e^{-kt} = A e^{-kt}$ so $T - K = \pm A e^{-kt}$ so

(absorbing the ± into the arbitrary constant "A") we get $T(t) = K + A e^{-kt}$. Note that $\lim_{t \to \infty} T(t) = K$ and the object eventually cools to the temperature of its surroundings. Note, too, that there are <u>three</u> constants in the solution: K, A and k so we'd need <u>three</u> pieces of information in order to determine T(t).

Example: Your professor is found dead in his office, after the final exam. The coroner arrives at 9:00 a.m. and records the body temperature as 30.1˚ C. One hour later she notes the body temperature has dropped to 29.2˚ C. When did the *accident* take place? (Assume Newton's law of cooling and a room temperature of 20˚ C. Note that normal body temperature is 37˚ C.)

Solution: The temperature at a time t hours after the accident is given by $T(t) = K + A e^{-kt}$. We know several things:

(1) the surroundings are at room temperature, so $\boxed{\mathbf{K = 20}}$ and

(2) initially T(0) = $\boxed{\mathbf{K + A = 37}}$ and

(3) if 9 a.m. is U hours after the accident then T(U) = $\boxed{\mathbf{K + A\, e^{-kU} = 30.1}}$ and

(4) one hour later T(U+1) = $\boxed{\mathbf{K + A\, e^{-k(U+1)} = 29.2}}$

Although we have four equations, that's necessary because we've had to introduce a fourth constant, U. We solve for the four constants K, A, k and U: K = 20 (from (1)) so A = 17 (from (2)) and $e^{-kU} = \frac{30.1\text{-}K}{A} = \frac{10.1}{17}$ (from (3)) and $e^{-k(U+1)} = \frac{29.2\text{-}K}{A} = \frac{9.2}{17}$ (from (4)) . Dividing the last two equations eliminates U and gives $e^{-k} = \frac{9.2/17}{10.1/17} = \frac{9.2}{10.1}$ so $e^{-kU} = \frac{10.1}{17}$ can be rewritten $(e^{-k})^U = \frac{10.1}{17}$ or $\left(\frac{9.2}{10.1}\right)^U = \frac{10.1}{17}$. Now take the *ln* of each side to get

$U\, ln\, \frac{9.2}{10.1} = ln\, \frac{10.1}{17}$ hence $U = \dfrac{ln\, \frac{10.1}{17}}{ln\, \frac{9.2}{10.1}} \approx 5.58$ so <u>this</u> many hours had passed before the temperature dropped from 37˚ to 30.1˚. That places the accident 5.58 hours <u>before</u> 9 a.m., namely at 3:25 a.m.

S: Why was the prof in his office at 3 in the morning?
P: Working, working, working ...
S: That'll be the day.

<u>LECTURE 3</u>

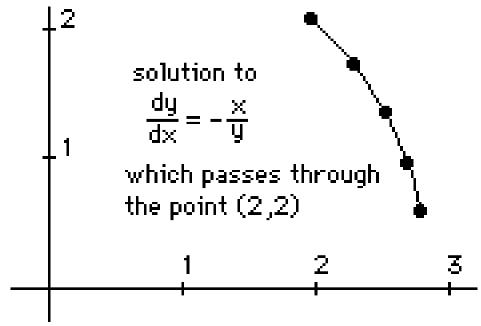
MORE ON DIFFERENTIAL EQUATIONS

<u>PS:</u>
S: This DE stuff is pretty neat, but what if the DE is something I can't solve. I mean, what if ...
P: Don't worry, most DEs can't be "solved" in the sense of being able to express the solutions in terms of known functions.
S: You've said that before ... I think.
P: You've got a good memory. Consider the world's simplest DE: $\frac{dy}{dx} = f(x)$. The solutions are $y = \int f(x)\, dx + C$ and if I can't integrate every function f(x) in terms of known functions then I can't solve the DE in terms of known functions. But that doesn't mean it doesn't have solutions, only that I can't express ...
S: ... the solutions in terms of known functions. Yeah, I can see that. But what about me? Am I supposed to be able to solve ...
P: There are only two types of DEs you'll see in this course ... well, maybe three types if you include the world's simplest, $\frac{dy}{dx} = f(x)$, which is really a problem in evaluating an integral. In fact, sometimes the solutions to DEs are actually called *integrals*, did you know that? And solving is sometimes called *integrating* a DE, did you know that?
S: Three types? Did you say <u>three</u>?
P: Pay attention. We'll talk about the next type in a minute but first let me say something about a DE of the form $\frac{dy}{dx}$ = *something* involving x and y which, except for very simple right-sides, we won't be able to solve in terms of known functions. Nevertheless, we <u>can</u> say something about these and we can actually sketch solutions and maybe that's all we need to know.
S: Huh?

Direction of Solutions

Example: Sketch the solution to the DE: $\frac{dy}{dx} = -\frac{x}{y}$ which passes through the point (2,2)

Solution: At the point (2,2) the solution has slope $\frac{dy}{dx} = -\frac{2}{2} = -1$, so we draw a tiny piece of the solution: a short line segment with slope -1. That brings us to another, nearby point and we compute the slope there as well, using $\frac{dy}{dx} = -\frac{x}{y}$ and, as before, we draw a tiny piece of the solution with this slope bringing us to another nearby point ... and we repeat the process, always moving in the direction given by the DE. The result will be a curve as shown ===>>>



In fact, the solution through (2,2) will move so that, wherever it finds itself, its direction is always $-\frac{x}{y}$. If you look at the solution you'll see that the slope becomes more and more negative at y decreases to zero ... as required by the DE: $\frac{dy}{dx} = -\frac{x}{y}$. In fact, you can just as easily sketch the solution through any chosen point and ...

S: Hold on! I can solve that DE ... it's separable! The solution is ... uh, I separate variables and get y dy = - x dx so I integrate and get $\int y\,dy = -\int x\,dx$ and that gives me $\frac{y^2}{2} = -\frac{x^2}{2} + C$ or I could just write $x^2 + y^2 = 2C$ and you can plainly see that the solutions are circles!

P: And what about the curve I sketched through (2,2). Doesn't it look like a circle?

S: Yeah, but you didn't have to do that. I mean ...

P: Okay, <u>you</u> sketch the solution to $\frac{dy}{dx} = x^2 + y^2$ which passes through (0,1).

S: But I can't solve the DE!

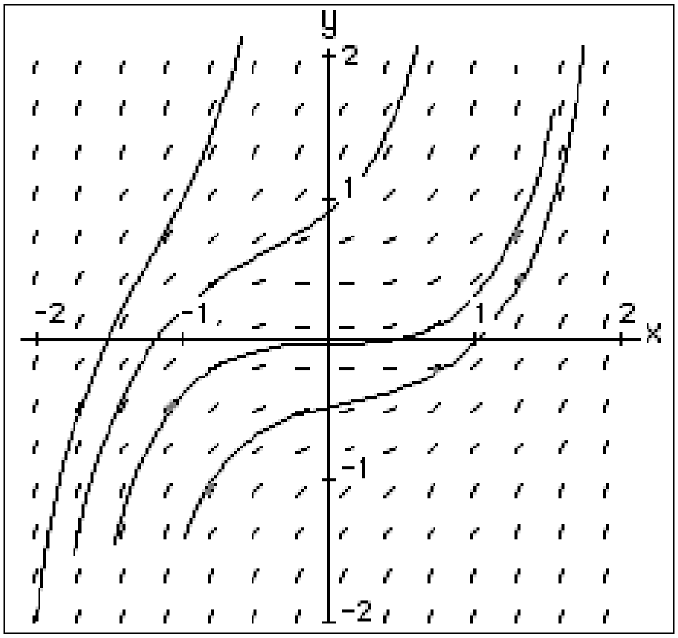
P: Precisely my point. This technique of "following the direction", as given by the DE, will work even if you can't "solve" the DE. Even if you <u>can</u> solve the DE it's sometimes easier and more instructive to "follow the direction" given by the DE ... which we can call the "DE Slope".

Given a DE of the form $\frac{dy}{dx} = f(x,y)$ = *a function of both x and y*, it is sometimes useful to fill the x-y plane with tiny pieces of solutions: pick a bunch of points (x,y) and draw a short straight line segment located at (x,y) with a slope given by f(x,y). In fact, from $\frac{dy}{dx} = f(x,y)$ we have, approximately, $\frac{\Delta y}{\Delta x} \approx f(x,y)$ so $\Delta y \approx f(x,y)\,\Delta x$ and <u>that</u> gives the change in y for a small change in x ... so we can "follow the DE".

This collection of "solution pieces" is sometimes called a DIRECTION FIELD, but we'll just call it **the DE PORTRAIT** since it gives at a glance the personality of the DE and its solutions. After all, a portrait is worth a thousand words ... or even a thousand equations.

Example: Sketch the DE PORTRAIT for $\frac{dy}{dx} = x^2 + y^2$

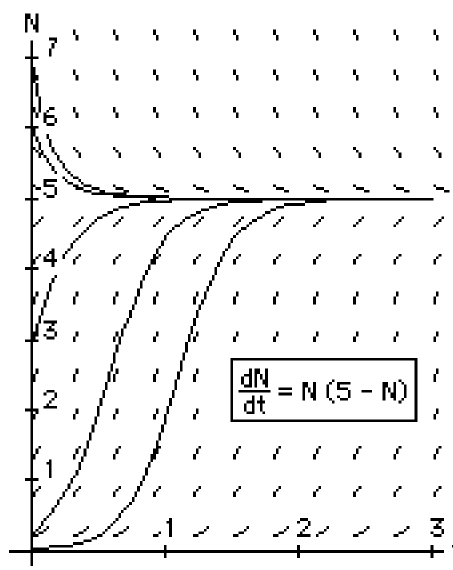
Solution: We can take a piece of the x-y plane, say the square $-2 \le x \le 2$, $-2 \le y \le 2$, and pick a gridwork of points in this square and at each point compute the value of x^2+y^2 and draw at that point a tiny line segment with <u>this</u> slope. This will give the portrait shown on the left, below:



Now we can pick a variety of starting points and sketch solutions to the DE through the selected point, being careful to follow the direction given by the portrait. This has been done above, at the right.

Example: Sketch the DE portrait for the Logistic Equation $\frac{dN}{dt} = N(5 - N)$ as well as some solutions.

Solution: Before we begin, let's try to predict the behaviour of solutions by inspecting the DE. Notice that, when $0 < N < 5$ we have $\frac{dN}{dt} > 0$ so the population N(t) increases --- the graph of solutions will have a positive slope. Also, when $N > 5$, note that $\frac{dN}{dt} < 0$ so N(t) decreases and that should be reflected in the graph of N(t) versus t. Note, too, that $N(t) = 5$ is an exact solution since $\frac{d}{dt}(5) = 0$ and $5(5 - 5) = 0$ so $N(t) = 5$ satisfies the DE. This solution is described graphically by a horizontal line.



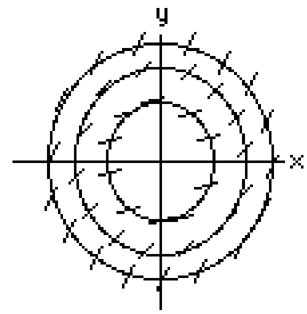
The Portrait is shown at the right, as well as several solutions. === >>>
Since the Logistic Equation is a model for population growth, we note with pleasure that for initial populations greater than 5, the population decreases and eventually approaches 5. Initial populations smaller than 5 grow and again have 5 as their limiting value. As mentioned earlier, 5 is called the "carrying capacity" and indicates a population in balance with its food supply and environment.
We might also note that small populations increase rapidly until they reach half of the carrying capacity (in this case, 2.5) then the rate of change of population decreases. i.e. $\frac{dN}{dt}$ decreases, meaning $\frac{d}{dt}\frac{dN}{dt}$ is negative, meaning the curve becomes concave down.
It should be clear that the DE Portrait really tells almost everything you'd want to know about the solutions to a DE.

S: But that's a lot of work! I mean, you have to pick dozens of points and calculate the slopes from the DE ... these DE Slopes ... and plot all those tiny lines. That's hard!

P: Not hard, just tedious. But if you're clever ... and I know you are ... you can sometimes simplify these calculations. For example, instead of picking a bunch of points (x,y) and determining the DE slope at each, you can pick the slope first, then determine the points which give that particular slope.

S: Huh?

P: Take, for example, $\frac{dy}{dx} = x^2 + y^2$. We could ask "Where are all the points with DE slope = 1?" and the answer would be "Everywhere where $x^2+y^2=1$", meaning every point on this circle has DE slope = 1 so you'd just have to draw this circle and place a bunch of tiny lines with slope = 1 on the circumference. Nice, eh? And the points with DE slope = 2? Everywhere on the circle $x^2+y^2=2$. And ...



S: A picture is worth ...

P: Here's a picture === >>>
If you delete the circles (which shouldn't be confused with the solutions to the DE!) then you'd be left with the DE Portrait.

LINEAR FIRST ORDER DEs:

Differential equations are classified according to the highest derivative that occurs.

- $\frac{dy}{dx} = x + y^2$ is a <u>first order</u> DE because the highest derivative is the first.
- $\frac{d^2x}{dt^2} + \frac{dx}{dt} + e^t x = 0$ is a <u>second order</u> DE since the highest derivative is the second.

In addition to the <u>order</u> of a DE, we can classify them as <u>LINEAR</u> and <u>NONLINEAR</u>. Linear 2nd order DEs can always be put into the form: *something* $\frac{d^2y}{dx^2}$ + *something* $\frac{dy}{dx}$ + *something* y = *something* where all those *something*s are functions of the independent variable x and NOT of the unknown function y or its derivatives (else it'd be called nonlinear). A SECOND ORDER LINEAR DE has the form: $A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)\,y = D(x)$ and a FIRST ORDER LINEAR DE has the form: $A(x)\frac{dy}{dx} + B(x)\,y = C(x)$ and so on. If a DE isn't linear, it's nonlinear. Usually, first order are easier to solve than second which are easier than third, etc. and linear are easier than nonlinear ... but that's not always true.

Example: Solve the second order nonlinear DE: $2xy\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx} = e^x$.

Solution: We recognize that the left-side is just the second derivative of xy^2 so we can rewrite the DE as: $\frac{d^2}{dx^2}(xy^2) = e^x$ hence we integrate each side to get $\frac{d}{dx}(xy^2) = e^x + A$ (adding a constant of integration, A) and (integrating once more) gives the solution as $xy^2 = e^x + Ax + B$ (adding another constant of integration, B).

S: What! We *recognize the left-side as the second derivative of xy^2*! I wouldn't recognize ...

P: I was just kidding. I actually invented the solution before I invented the DE. I just took $xy^2 = e^x + Ax + B$ and differentiated each side twice. But see how easy it was to solve? And it's second order <u>and</u> nonlinear! On the other hand, what appears as a simple first order DE, $\frac{dy}{dx} = x + y^2$, is impossible to solve in terms of well-known functions. But if a first order DE happens to be <u>linear</u>, then we can *always* solve it.

<u>First order linear</u> DEs always have the form $A(x)\frac{dy}{dx} + B(x)\,y = C(x)$ but before we proceed we put it into "standard" form by dividing through by A(x) giving $\frac{dy}{dx} + \frac{B(x)}{A(x)}\,y = \frac{C(x)}{A(x)}$ and we'll let $\frac{B(x)}{A(x)} = P(x)$ and $\frac{C(x)}{A(x)}$

$= Q(x)$ so our first order linear DE now has the "standard" form: $\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$. We illustrate the method of solution with an example:

Example: Solve $\frac{dy}{dx} + \frac{1}{x}\,y = \frac{e^{-2x}}{x}$.

Solution: Multiply through by x to get $x\frac{dy}{dx} + y = e^{-2x}$. Now recognize the left-side as the first derivative of xy, so we rewrite our DE as: $\frac{d}{dx}(xy) = e^{-2x}$ and then we integrate each side to get: $xy = -\frac{1}{2}\,e^{-2x} + C$ (adding a constant of integration, C) so we have our solution and, in this example, we could also solve for $y = \frac{-\frac{1}{2}e^{-2x} + C}{x}$

S: There you go again ... recognizing the left-side as the derivative of something! How would *I* recognize that? And how did you know to multiply by x? And why did you ...
P: Okay, pay attention. Here's the technique ... and it's very clever.

Starting with $\frac{dy}{dx} + P(x)\,y = Q(x)$ (the linear first order DE in "standard form") we want to multiply by *something* so that the resulting left-side is precisely the derivative of (*something* times y). That is, we multiply by, say, $\mu(x)$, giving $\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y = \mu(x)\,Q(x)$ and insist that the left-side $\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y$ is exactly the derivative: $\frac{d}{dx}(\mu(x)\,y)$. But $\frac{d}{dx}(\mu(x)\,y) = \mu(x)\frac{dy}{dx} + \frac{d\mu(x)}{dx}\,y$ so we'd stare at $\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y$ and see that we'd need $\frac{d\mu(x)}{dx}\,y = \mu(x)\,P(x)\,y$ and that means that $\mu(x)$ must be chosen so that $\frac{d\mu(x)}{dx} = \mu(x)\,P(x)$ which, although it's a DE to solve for $\mu(x)$, it's a *separable* DE. We'd get: $\frac{d\mu}{\mu} = P dx$ so that $\int\frac{d\mu}{\mu} = \int P dx$ so that $ln\ |\mu| = \int P\,dx$ and that means we can choose $\mu(x) = e^{\int P(x)dx}$. After multiplying by this function, $\mu(x)$, the DE can be rewritten $\frac{d}{dx}(\mu y) = \mu Q$ so we'd just integrate each side: $\mu y = \int \mu Q\,dx$ and add an arbitrary constant ... and we'd be finished.

> The linear first order DE: $\frac{dy}{dx} + P(x)\,y = Q(x)$ can be solved by first multiplying by
> $$\mu(x) = e^{\int P(x)dx}.$$
> It then becomes $\frac{d}{dx}(\mu(x)\,y) = \mu(x)\,Q(x)$ and can be integrated directly.

S: Hold on! First, you forgot to add an arbitrary constant when you integrated P. You <u>should</u> have written $ln\ |\mu| = \int P dx + C$. Secondly, you ...

P: No. I only want *some* $\mu(x)$ that works ... *one* will do. I don't need every possible solution to the DE: $\frac{d\mu}{dx} = \mu P$, just one. Besides, if I *did* add a constant I'd get $\mu(x) = e^{\int P(x)dx+C} = e^C\,e^{\int P(x)dx}$ so it'd just be multiplying my $\mu(x)$ by some constant, e^C, and hence I'd be multiplying the original DE by such a constant. No need to do that. See?
S: Let me try one:

P: Okay, solve: $x^2 \frac{dy}{dx} + x\,y = x\,e^{-2x}$.

S: Right! First I'd find $\mu = e^{\int P(x)dx}$ and P(x) = x (the coefficient of the y-term, right?) so $\mu = e^{\int x dx} = e^{x^2/2}$ then I'd multiply the DE by this guy and get $x^2 e^{x^2/2} \frac{dy}{dx} + x\,e^{x^2/2}\,y = x\,e^{-2x}$ (which looks pretty awful) and the left-side is exactly $\frac{d}{dx}\,\mu y$ so I'd rewrite the DE as $\frac{d}{dx}\left(e^{x^2/2}y\right) = x\,e^{-2x}$ then I'd integrate each side ... hey! That's tough, right? I don't know how to integrate ...

P: And no wonder ... you made so many mistakes. First off you forgot to put the DE into "standard" form (where the coefficient of $\frac{dy}{dx}$ is just "1") . That means you'd first have to divide by x^2 and that'd give you $\frac{dy}{dx} + \frac{1}{x}\,y = \frac{e^{-2x}}{x}$ which you should recognize because I've just done that one! Besides that, you said the *left-side is exactly* $\frac{d}{dx}\,\mu y$ and you didn't even bother to check that! In fact, had you checked to see if $x^2 e^{x^2/2} \frac{dy}{dx} + x\,e^{x^2/2}\,y$ was really $\frac{d}{dx}\left(e^{x^2/2}y\right)$ you'd find it wasn't, so you'd know that you'd made a mistake. So here's a piece of advice: after you *think* you've found μ(x), multiply the DE by this μ(x) (and don't forget to multiply the right-side too ... and you DID forget ... and that's another mistake!) then CHECK TO SEE IF THE LEFT-SIDE IS $\frac{d}{dx}\left(\mu(x)\,y\right)$. If it is, then you've got yourself a correct integrating factor.

S: A correct what!

P: Oh, I forgot to tell you. The function μ(x) is called an INTEGRATING FACTOR because it allows us to "integrate" the DE. Did I tell you that? Solving a DE is sometimes called ...

S: Yeah ... *integrating the DE* ... I know, I know. Give me another one.

P: Okay, solve $\sin x \frac{dy}{dx} + (\cos x)\,y = \tan x$.

S: Right! First I divide through by sin x and get $\frac{dy}{dx} + \frac{\cos x}{\sin x}\,y = \frac{\tan x}{\sin x}$ (... that looks pretty bad ...) then I'd pick out P(x) = $\frac{\cos x}{\sin x}$ (that's the coefficient of y) and I'd integrate it to get $\int P(x)\,dx = \int \frac{\cos x}{\sin x}\,dx$ ... ?!$# Can I do that? ... uh, yes, I'd let u = sin x so that $du = \frac{du}{dx}\,dx = \cos x\,dx$ and the integral turns into $\int \frac{du}{u} = \ln|u|$ and I'd forget the "+C" because ... uh, can't remember why ... then I'd have $\int P(x)\,dx = \ln|\sin x|$ so I'd multiply through by this and ...

P: You'd what?!

S: Oh, forgot ... first I'd write $\mu = e^{\int P(x)dx} = e^{\ln|\sin x|}$ which is ... uh, don't tell me ... it's just | sin x | ... that's one of those log things ... then I'd multiply the DE by this guy ... where's the DE? Oh yeah, I'd multiply $\sin x \frac{dy}{dx} + \cos x\,y = \tan x$ by |sin x| and I'd get ...

P: You'd what?!

S: Oh, sorry, I need the other DE, right? I mean, I'd multiply $\frac{dy}{dx} + \frac{\cos x}{\sin x}\,y = \frac{\tan x}{\sin x}$ by |sin x| and I'd get ... wow, that ain't easy, is it? Can't I just drop the absolute value thing. Sure, why not. I just multiply by sin x and I'd change the DE into ... uh, it becomes $\sin x \frac{dy}{dx} + \cos x\,y = \tan x$. Hey! That's what I started with! What's happening here?

P: And is the left-side exactly the derivative of ... of what?

S: Oh yeah, I know. It should be the derivative of μ y or (sin x) y and that's $(\sin x)\frac{dy}{dx} + \cos x\,y$ and it IS! How'm I doin' boss?

P: Keep going, you're not finished.

S: Okay, the DE after I've fixed it up is $\frac{d}{dx}\left((\sin x)\,y\right) = \tan x$ ... and if I were smart I'd have recognized it right off the bat ... so I'd integrate each side and I'd get $(\sin x)y = \int \tan x\,dx = \ln|\sec x|$ ... I actually remembered that one ... and I'd be finished, right? I mean, that's the solution, right?

P: Wrong. Where's your arbitrary constant?

S: I decided to drop it. You did, remember? You said you only wanted one solution, *any* one, so you just upped and left out the

+C.

P: That's because I wanted only *one* integrating factor, μ(x). For the DE I want ALL solutions, not just one.

S: Well, I have *one* for you and it's $y = \frac{\ln|\sec x|}{\sin x}$. Like it?

P: It's called a "particular solution" as opposed to the "general solution", and yes, I like it very much, but is it a solution? Remember, you decided to drop the absolute value sign. How would you check it?

S: I haven't the foggiest. Oh, wait, I just plug it in, right? I mean, I just put $y = \frac{\ln|\sec x|}{\sin x}$ and see if $\sin x \frac{dy}{dx} + \cos x\, y = \tan x$. Okay, I get $\frac{dy}{dx} = \frac{\sin x \tan x - \ln|\sec x| \cos x}{\sin^2 x}$ and I'd substitute into the DE and I'd get $\sin x \left(\frac{\sin x \tan x - \ln|\sec x| \cos x}{\sin^2 x}\right) + \cos x \left(\frac{\ln|\sec x|}{\sin x}\right)$ on the left-side and this had better be $\tan x$ when the smoke has cleared ... uh, is it? I leave it as an exercise for the prof.

When solving DEs you often resort to a "Table of Integrals". In the following, you may select from the following table (which omits the +C):

$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$	$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left\lvert \frac{a + x}{a - x} \right\rvert$
$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$	$\int \frac{dx}{(a + x)(b + x)} = \frac{1}{b - a} \ln \left\lvert \frac{a + x}{b + x} \right\rvert$	

Example: An object stands in a field and heats and cools as the day progresses. If the air temperature varies according to $K = 20 + 5 \sin \frac{\pi t}{12}$ degrees Celsius (hence varies from 15˚C to 25˚C over a 24 hour period), determine the temperature of the object as a function of time t (measured in hours) assuming Newton's Law of Cooling.

Solution: If T(t) is the temperature of the object at time t, then Newton's Law states: $\frac{dT}{dt} = -k (T - K)$ which would be separable if the air temperature K were constant, but it isn't! In fact, it's linear first order so we write it in "standard form": $\frac{dT}{dt} + k T = k (20 + 5 \sin \frac{\pi t}{12})$ where we've substituted for K. An integrating factor is $\mu(t) = e^{\int k\, dt} = e^{kt}$ and after multiplication by e^{kt} the DE becomes $e^{kt} \frac{dT}{dt} + k e^{kt} T = k (20 e^{kt} + 5 e^{kt} \sin \frac{\pi t}{12})$ and we check to see that the left-side is exactly $\frac{d}{dt} \mu T = \frac{d}{dt} e^{kt} T$ which it is (and we're happy). Hence we can rewrite the DE as:

$\frac{d}{dt}(e^{kt} T) = k (20 e^{kt} + 5 e^{kt} \sin \frac{\pi t}{12})$ and integrate each side to get (with the help of the table of integrals above):

$$e^{kt} T = 20 e^{kt} + 5k \frac{e^{kt}}{k^2+(\pi/12)^2} \left(k \sin \frac{\pi t}{12} - \frac{\pi}{12} \cos \frac{\pi t}{12}\right) + C$$ or, more simply

$$T(t) = 20 + \frac{5k}{k^2+(\pi/12)^2} \left(k \sin \frac{\pi t}{12} - \frac{\pi}{12} \cos \frac{\pi t}{12}\right) + C e^{-kt}.$$

S: Are you finished?

P: Sure. I wanted to get the temperature and I got it.

S: But what about the +C? Who is C? You gotta find C!

P: It's impossible without more information. After all, suppose the object was originally at 200˚, then the solution would be quite different than if it were originally at - 150˚. See? The constant C will depend upon ...

S: Okay, put the object in the sun at ... uh, 100˚ Celsius. Then what?

P: Then at t = 0 I'd put T = 100 and get $100 = 20 + \frac{5k}{k^2+(\pi/12)^2} \left(0 - \frac{\pi}{12}\right) + C$ and that'd give me C, see? But actually it matters little because whatever C is, the term $C e^{-kt} \to 0$ and the temperature becomes just

$T(t) = 20 + \frac{5k}{k^2+(\pi/12)^2}\left(k \sin\frac{\pi t}{12} - \frac{\pi}{12}\cos\frac{\pi t}{12}\right)$ and you get this variation in temperature of the object no matter what temperature it started with ... if you wait long enough.

S: Seems you have to wait forever for $C\,e^{-kt} \to 0$, right?

P: Oh well, the math is just an approximation anyway. When this term is just a fraction of a degree we can ignore it ... and that might just be an hour or two.

S: *The math is just an approximation*? You always told me that ...

P: Look, how good is Newton's Law of Cooling in this situation? And what happens if the object is in the shade, under a tree, then the sun hits it, so the temperature isn't proportional to the *air* temperature as Newton's Law assumes, but the object gains heat directly from the sun. Does the math know that? No! And what if there's a wind which cools the object and what if we were to take into account the heating and cooling from the ground and what if ...

S: Okay, okay, but if the math is only an approximation then what good is it? I can just say the temperature of the object is about 20˚ and that'd be an approximation too, right? And I wouldn't need anybody's law of cooling for that.

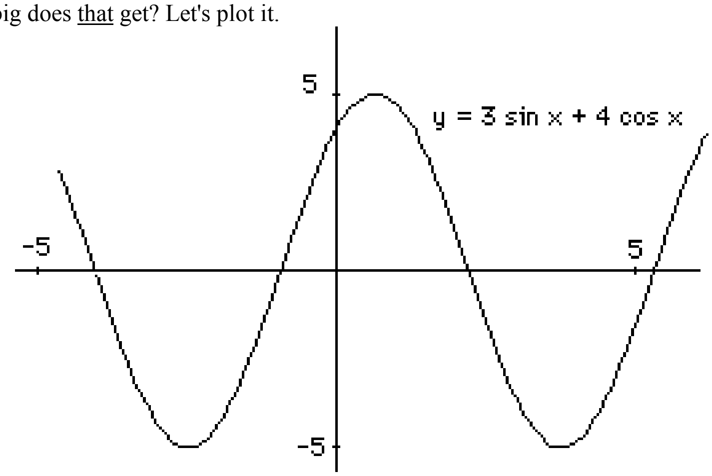
P: You're right, of course, but I'd bet a tidy sum that *my* approximation is <u>much</u> better than yours ... and that's what the math is doing for us. Remember, we have to use some cerebral prowess as well as turning a mathematical crank. When we counted the number of clams we got 10^{19} or something like that. It was a ridiculous answer and we should *know* that ... so our DE provided a poor description of the population growth ... so we changed the DE to the logistic equation and got a much more reasonable answer, but it was <u>still</u> an estimate after all. If you didn't care about how good the estimate was, you could certainly say "I think there'd be roughly 100 million clams" and leave it at that.

S: What about the object in the field? Do you think you've got a good estimate of the temperature?

P: Let's compute some values. Our temperature was $T(t) = 20 + \frac{5k}{k^2+(\pi/12)^2}\left(k \sin\frac{\pi t}{12} - \frac{\pi}{12}\cos\frac{\pi t}{12}\right)$ ... after some time when the effects of the initial temperature, contained in the term $C\,e^{-kt}$, die out. Okay, how big does T get, and how small? I wouldn't trust this result if T varied from - 10˚ to over 100˚. So tell me, how big does T get?

S: Are you kidding? That's tough! I mean, how big does $\left(k \sin\frac{\pi t}{12} - \frac{\pi}{12}\cos\frac{\pi t}{12}\right)$ get?

P: It only looks tough because of all those symbols. Let's make it simpler, in fact we'll take a specific example, say $y = 3 \sin x + 4 \cos x$. How big does <u>that</u> get? Let's plot it.



Does it look familiar?

S: It gets as big as 5!

P: And it's looks like a sine curve, but shifted. It could be $A \sin (x + B)$ if I could pick the numbers A and B correctly.

S: But you just wanted to know how big it is ... and I already told you. It gets as big as 5!

P: Pay attention. We're going to learn something exciting here. We'd like to have $3 \sin x + 4 \cos x = A \sin (x + B)$, if we can. The left-side has separate sines and cosines so we'd want that on the right too, so we'd write
$A \sin (x+B) = A\left(\sin x \cos B + \cos x \sin B\right)$ using a well-known trig formula ...

S: Well-known to you maybe, but ...

P: ... so we now have to find numbers A and B such that $3 \sin x + 4 \cos x = (A \cos B) \sin x + (A \sin B) \cos x$ so we have <u>two</u> equations in <u>two</u> unknowns, namely: $A \cos B = 3$ and $A \sin B = 4$. To solve we could divide and eliminate A getting $\tan B = \frac{4}{3}$ and that'd give us B (if we wanted it, but we don't because we're really more interested in A, the *amplitude* of $A \sin (x+B)$) so to find A we can square and add: $(A \cos B)^2 + (A \sin B)^2 = 3^2 + 4^2$ because that'd get rid of B (since $\cos^2 B + \sin^2 B = 1$). We then get $A^2 = 3^2 + 4^2$ so that $A = \sqrt{3^2+4^2} = \sqrt{25}$ and ...

S: See! I told you it was 5 but you weren't paying attention!

P: I'm much more interested in the result of this analysis. It says that the expression 3 sin x + 4 cos x can be written in the form $\sqrt{3^2+4^2}$ sin (x+B) and that means it gets as large as $\sqrt{3^2+4^2}$ and as small as $-\sqrt{3^2+4^2}$ and, in general, ...

S: You keep saying $\sqrt{3^2+4^2}$ but I'm telling you it's 5! Aren't you any good at arithmetic?

P: I keep the 3 and 4 separate so I can see what happens in general ... and <u>this</u> is what happens:

$$P \sin \omega t + Q \cos \omega t = \sqrt{P^2 + Q^2} \sin(\omega t + B) \text{ for some number B}$$

Now <u>that's</u> a nice trig identity!

S: How about the thing that's waiting in the field? How hot does it get?

P: <u>You</u> figure it out.

S: Okay, I just need to find P and Q and they're ... uh, we were asking how big $\left(k \sin \frac{\pi t}{12} - \frac{\pi}{12} \cos \frac{\pi t}{12}\right)$ got, so P = k and Q = $-\frac{\pi}{12}$ so it gets as big as $\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}$ ... but I don't know k, do I?

P: No, but let's be careful. The temperature was $T(t) = 20 + \frac{5k}{k^2+(\pi/12)^2}\left(k \sin \frac{\pi t}{12} - \frac{\pi}{12} \cos \frac{\pi t}{12}\right)$ and so gets as large as $20 + \frac{5k}{k^2+(\pi/12)^2}\left(\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}\right)$ which is the same as $20 + \frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$. To find k we'd need to have more ...

S: Yeah, I know, you need more information . What more do you want?

P: Just one temperature reading should do it because there's just one constant we need, namely k. But we needn't go out with our thermometer because we just wanted to know if our expression for T(t) should be trusted. Notice that it varies sinusoidally between $20 + \frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$ and $20 - \frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$, much like the air temperature does. Remember, the air temperature was $K = 20 + 5 \sin \frac{\pi t}{12}$ and varied from 20 + 5 to 20 - 5. In fact the term $\frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$ isn't even as large as 5, no matter *what* value k has.

S: Huh?

P: Don't you see? $\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}$ is larger than $\sqrt{k^2+0}$ which is just k, so $\frac{k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$ is smaller than 1 so the temperature variations of $\frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}}$ are actually smaller than 5 (1) = 5. Nice, eh? I think we may have something here. Doesn't it give you some faith in the analysis?

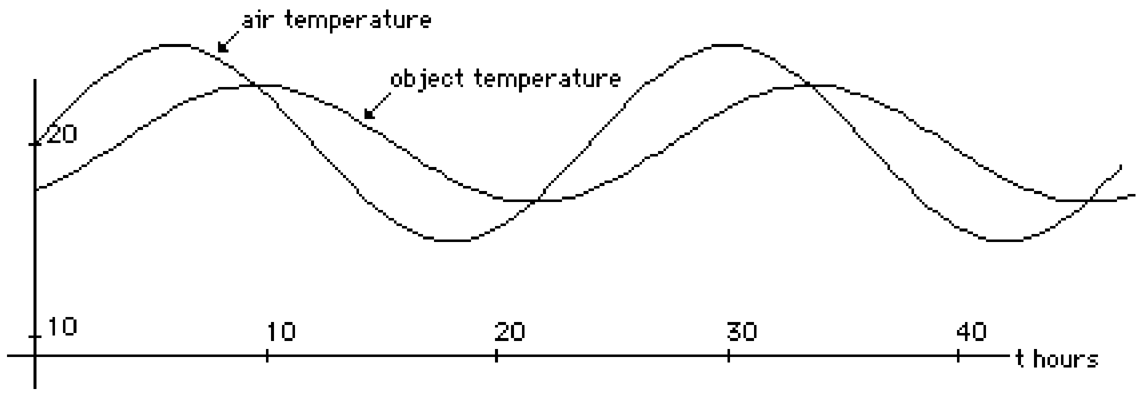
S: Say, do I have to know this for the final exam?

P: Before we leave this, let's pretend we've measured the maximum temperature of the object in the field and it's 23° Celsius ... and we assume the object has been sitting there for some time so we can ignore the Ce^{-kt} term in the solution. Then $20 + \frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}} = 23$ so we can solve for k and it's ... uh, let's see, it's ...

S: Let me do it. I'd get $\frac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}} = 3$ so I'd square and get $\frac{25k^2}{k^2 + \left(\frac{\pi}{12}\right)^2} = 9$ so

$25k^2 = 9k^2 + 9\left(\frac{\pi}{12}\right)^2$ so $16k^2 = 9\left(\frac{\pi}{12}\right)^2$ and $k = \frac{3}{4}\frac{\pi}{12} = \frac{\pi}{16}$. Good?

P: Good. Now let me plot the temperature of the object <u>and</u> the temperature of the air, but first let me stick in the value $k = \frac{\pi}{16}$ and get the object temperature as $T(t) = 20 + \frac{9}{5} \sin \frac{\pi t}{12} - \frac{12}{5} \cos \frac{\pi t}{12}$:



Nice, eh? The object lags behind, in temperature, by ... it looks like about 4 hours. Now that's something I'd have some faith in, this lagging business ... and the object has temperature swings which are smaller than the surrounding air ... and the variations have a 24 hour period just like the air and ...

S: Okay, I get it, you want me to be impressed with what the math is saying. So, I'm impressed. But I'll tell you one thing: you say the object has temperature swings which are smaller than the air, but you assumed that the maximum temperature of the object was 23˚ so no wonder! If you had assumed the maximum object temperature was 30˚ then ...

P: Try it!

S: Okay, I'd want $20 + \dfrac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}} = 30$ which means I'd get $\dfrac{5k}{\sqrt{k^2 + \left(\frac{\pi}{12}\right)^2}} = 10$ so I'd divide by 5 and square and get $\dfrac{k^2}{k^2 + \left(\frac{\pi}{12}\right)^2} = 4$ so $k^2 = -\frac{4}{3}\left(\frac{\pi}{12}\right)^2$ and ... uh, how's that possible? I mean, k^2 is negative.

P: See how smart the math is? It's impossible for the object to have a temperature greater than the maximum air temperature ... and the math is telling you that.

S: It is? Do you believe it?

P: I'm not sure. It's a prediction based upon this particular set of assumptions ... like Newton's law of cooling, for example. We'd have to go out into the field and do some measuring to see if the math is ...

S: Aha! The math may be no good!

P: No, the math is always good ... it only does what it's told. You say "assume Newton's Law of Cooling" and it does. It's the physicist or engineer or biologist who may be no good at making reasonable assumptions hence generating a good MATHEMATICAL MODEL ... and that's what these DEs are ... just so-called mathematical models of how things behave and they or may not be any good.

Example: Water flows into lake Ontario at the rate of A *metres³/day* (from rivers and rain, etc. as well as from liquid industrial waste) and this water has an average pollution concentration of B *kg/metres³*. Water (mixed with pollutants) is also withdrawn at the rate of C *metres³/day*. Describe, via a differential equation, the amount of pollutants at time t days (after measurements begin).

Solution: If P kg is the amount of pollutants in the water at time t, then the rate of change, measured in kg/day, is $\boxed{\dfrac{dP}{dt}\ kg/day = (kg/day\ \text{IN}) - (kg/day\ \text{OUT})}$. But water enters at A *metres³/day* and has a pollutant concentration of B *kg/metres³* so: $(kg/day\ \text{IN}) = \left(A\,\dfrac{metres^3}{day}\right) \times \left(B\,\dfrac{kg}{metres^3}\right) = AB\,\dfrac{kg}{day}$. Further, if the polluted water leaves at $C\,\dfrac{metres^3}{day}$ we can find the (*kg/day* OUT) if we know the *kg/metres³* concentration. If Lake Ontario has a volume of V *metres³* and the lake contains P *kg* of pollutants, the average concentration if $\dfrac{P\ kg}{V\ metres^3} = \dfrac{P}{V}\ kg/m^3$. However, V may change with time. In fact, its rate of change is

$$\boxed{\frac{dV}{dt}\ metres^3/day = (metres^3/day\ \text{IN}) - (metres^3/day\ \text{OUT})} = (A)\ m^3/day - (C)\ m^3/day$$

and, unless A = C the volume will change. We solve this world's simplest DE for V(t): $\dfrac{dV}{dt} = A - C$ hence $V = (A - C)\,t + constant$

(where the *constant* of integration won't be called C else we'd get it confused with the C m^3/day!). If the lake initially (when measurements begin) has V_0 *metres*^{3} of water (mixed with pollutants) then $V_0 = 0 +$ *constant* and we conclude that the *constant* $= V_0$ and the volume of the lake is $V(t)=(A-C)\,t + V_0$ *metres*^{3} at time t, so the average concentration of pollutants is $\frac{P(t)}{V(t)} = \frac{P(t)}{(A-C)\,t+V_0}$ and we finally have our DE:

$$\boxed{\frac{dP(t)}{dt} = AB - \frac{C\,P(t)}{(A-C)\,t+V_0}}$$

which (surprise!) is a linear first order DE, easily recognizable if we write it as:

$$\frac{dP}{dt} + \left(\frac{C}{(A-C)\,t+V_0}\right) P = AB.$$

S: I hope I don't see that on the final exam! Say, do you think we've ... uh, *you've* done a good job in finding a reasonable mathematical model?

P: Not really, but as a first try it'll give me an estimate which will probably be in the ball park. You see, I've used $(kg/day\ \text{OUT}) = (C\ m^3/day)\left(\frac{P}{V}\ kg/m^3\right)$ hence I've assumed an "average" pollutant concentration of $\frac{\text{total kilograms of pollutants}}{\text{total volume of lake}} = \frac{P}{V}$ which is like saying that the places where lake water is removed are places where the pollutant concentration is average. Not likely. Besides, I've assumed that the only source of pollutants is from water which enters the lake ... maybe there are other sources. Besides, I've assumed that every day is like every other day whereas I might improve the model by considering the A, B, C etc. to change with time. Besides ...

S: Okay, I get the idea.

LECTURE 4

SEQUENCES AND SERIES

Sequences

PS:

P: What's this series add up to? $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

S: I hope it's a geometric series ... I have a formula for that ... let's see, $\frac{1/2}{1} = \frac{1/4}{1/2} = \frac{1/8}{1/4}$ so the ratios are all the same so it is a geometric series, so it adds up to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$, right?

P: Why 2? Why not 3?
S: I have a formula, that's why!
P: If you told somebody it "adds up to 2" you'd mean ... what?
S: I'd mean ... if you added up all the terms you'd get the number 2.
P: Can you add up all the terms? Is that possible? Remember, there are an infinite number of terms.
S: You're trying to tell me something, right?
P: How about the series: $0 + 0 + 0 + 0 + \ldots$?
S: Easy! It adds up to 0.
P: Why? There are an infinite number of terms, that's ∞, and each has the value 0, so shouldn't it add up to (∞)(0) ... and just what does that mean?
S: Okay, you're trying to tell me we need a definition ... a *precise* definition, of "adding up a series", right?

Consider the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ where the first term is $a = 1$ and the *common ratio* is $r = \frac{1}{2}$ and we want a definition of "the sum of an infinite series" so that, for this series, the sum would be $\frac{a}{1-r} = 2$. If we added the numbers on a calculator the sequence of numbers appearing in the display window of the calculator would be 1 then $1 + \frac{1}{2} = 1.5$ then $1 + \frac{1}{2} + \frac{1}{4} = 1.75$ then $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$ and so on. If we expected to get a sum after adding an infinite number of terms we'd expect these numbers appearing in the calculator display window to approach some limiting value ... in this case the number "2". That is the basis for our definition:

> For the infinite series $a_1 + a_2 + a_3 + a_4 + \ldots$ we construct the sequence of PARTIAL SUMS
>
> $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, etc. and if $\lim_{n\to\infty} S_n = L$, then we say the infinite series "sums to L".

In other words: as we add the terms on a calculator, the numbers appearing in the calculator display are the "partial sums" and they should have a limiting value if the infinite series is to have a SUM. If they don't, the infinite series has <u>no</u> sum.

For the geometric series above, where $a = 1$ and $r = \frac{1}{2}$, it's fortunate that we have a formula for the sum of n terms: it's given by the formula $S_n = a\frac{1 - r^n}{1 - r}$ = which can be written $S_n = 2\left(1 - \frac{1}{2^n}\right)$. Clearly $\lim_{n\to\infty} S_n = 2$ (since $\lim_{n\to\infty} \frac{1}{2^n} = 0$) and we're happy that our definition agrees with this formula. For the series $0 + 0 + 0 + \ldots$ the PARTIAL SUMS are $S_1 = 0$ and $S_2 = 0 + 0 = 0$ and $S_3 = 0 + 0 + 0 = 0$ and, indeed, $S_n = 0$ no matter how many terms we add. Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 0 = 0$ and ...

S: But that's exactly what <u>I</u> said!
P: Aah, but now you can <u>prove</u> it! Nobody can argue that "the sum of the series is $(\infty)(0)$ hence doesn't exist".
S: You called this lecture "Sequences". Is that what we're studying? The "sequence" of partial sums? And is that so you can invent a definition for the "sum of an infinite series" ... using this "sequence" stuff?
P: Partially, but sequences occur from time to time without being associated with an infinite series, so they're worthwhile in their own right. See?
S: No.
P: Pay attention.

Examples:

- The <u>sequence</u> $A_n = A_0\left(1 + \frac{i}{100}\right)^n$ is the amount of money accumulated after n years, if $\$A_0$ is initially invested at i% per annum.
- If bacteria grows at 2% per day, then the <u>sequence</u> $B_n = B_0(1.02)^n$ gives the amount after n days.
- "Newton's method" provides the scheme $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with $n = 1, 2, 3, \ldots$ and x_1 given, which generates a <u>sequence</u> which (sometimes) converges to a root of $f(x) = 0$.
- In investigating the growth of a rabbit population, Fibonacci* generated a <u>sequence</u> $F_1, F_2, F_3, \ldots$ (called the "Fibonacci numbers") satisfying $F_{n+2} = F_{n+1} + F_n$ (each being the sum of the preceding two). They are: 1, 1, 2, 3, 5, 8, 13, ...
- The <u>sequence</u> of numbers $a_n = \left(1 + \frac{1}{n}\right)^n$ converges to the number e = 2.71828...

S: Hey! That's twice you've used the word "converges". What does ... ?
P: Okay, here's what we mean:

---

* This sequence was first studied by the Italian mathematician, Leonardo of Pisa (also known as Fibonacci ... one of the most brilliant pre-Renaissance mathematicians) around 1200 A.D.

> If $\lim_{n\to\infty} S_n = L$ we say that the sequence $\{S_n\}$ "converges to L".

We might say "this series converges" and "that one diverges" or maybe "here's a convergent series" and "there's a divergent series".

S: But that's just like saying that the limit exists, isn't it?

P: Yes, but it's more descriptive don't you think? If we're adding up the terms of an infinite series and we get the partial sums S_1 then S_2 then S_3 and so on, it's nice to think of them as "converging" to some limiting value. It's a nice terminology, don't you think?

S: No ... it's just more for me to remember. Another thing, do I know that $a_n = \left(1+\frac{1}{n}\right)^n$ converges to e? I mean, did you ever say that before? I mean ...

P: Let's do it. Sometimes it's difficult to establish the convergence of a sequence. For example, how would one prove that the sequence of Newton iterates, x_n, actually converge to a root of f(x) = 0? Not easy! But this one <u>is</u> easy. We just take the limit of $\left(1+\frac{1}{n}\right)^n$ as $n\to\infty$. First write $y = \left(1+\frac{1}{x}\right)^x$ so $\ln y = x \ln\left(1+\frac{1}{x}\right)$ and note that this has the form $(\infty)\ln(1) = (\infty)(0)$ as $x\to\infty$, so we couldn't use l'Hopital's rule so we rewrite it as $\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$ so it now has the requisite $\frac{0}{0}$ form, so now we can apply l'Hopital, differentiating both numerator and denominator so the new ratio is $\dfrac{\frac{1}{1+1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$ which is just $\dfrac{1}{1+\frac{1}{x}}$ and <u>now</u> we let $x\to\infty$ and get "1", see?

S: But you said the limit was "e"!

P: Oh, I forgot ... since $\ln y = x \ln\left(1+\frac{1}{x}\right)$ we've actually calculated the limit of $\ln y$, so y itself has a limit of $e^1 = e$. Nice, eh?

S: Shouldn't you say that "y converges to 1"?

P: Good idea, let's do that.

S: Another thing. I notice that you changed from "n" to "x". I mean, you actually calculated the limit of $\left(1+\frac{1}{x}\right)^x$ rather than the limit of $\left(1+\frac{1}{n}\right)^n$. Was that necessary?

P: Well, "n" is an integer and I didn't like the idea of differentiating with respect to an integer variable. After all, the definition of the derivative (remember?) is the limit of $\frac{\Delta y}{\Delta n}$ as $\Delta n\to 0$ and it's hard to imagine Δn as a tiny increment in a variable which can only have integer values. Can an integer change by, say, .001 and still be an integer? Hardly. So I changed to x which is a nice name for a continuous variable and then I ...

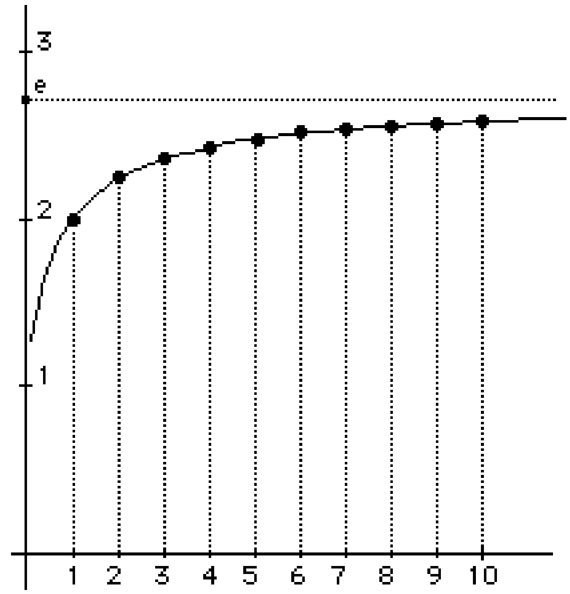
S: Come on, you just changed the problem, didn't you? You wanted the limit of one thing so you changed it and found the limit of something else. That's cheating, isn't it?

P: Pay attention. I'll plot a point $\left(1+\frac{1}{n}\right)^n$ for various values of the integer n, then I'll plot $\left(1+\frac{1}{x}\right)^x$ and you'll see that the curve passes through each point ... what else? To find the limit of $\left(1+\frac{1}{n}\right)^n$ as n–>∞, we just have to find the limit of $\left(1+\frac{1}{x}\right)^x$ as x–>∞ ... and that's just what we did.

S: But you could have kept the "n" ... you didn't have to change it to "x", did you? I mean, you'd get the same answer, right?

P: It just makes me feel better to call the variable "x". I really get nervous differentiating with respect to a variable which ...

S: Yeah, I know, with respect to an integer. Okay, I wouldn't want you to get nervous. Can we go on?



Plot of the sequence $\left(1+\frac{1}{n}\right)^n$

SERIES

Should we run into an infinite series, $a_1 + a_2 + a_3 + \ldots$, and it should happen to be a geometric series, then we'd just check to see if the common ratio r has an absolute value less than 1 (i.e. $|r| < 1$) and if so, we can "sum the series" using $\frac{a}{1-r}$. That is, the infinite series converges to $\frac{a}{1-r}$. Mother Nature is rarely so accommodating. You're more likely to run into a series for which you do NOT have a formula for the sum ... so you'd just have to start adding terms and pray that the partial sums have some limiting value.

Example: Calculate the sum of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$

Solution: We'll ask **MAPLE** to do the calculations. We'll start off by asking for 6 digit accuracy, then we'll define the terms: $t(k)=\frac{1}{k}$, then we'll find the **sum** for k = 1 to 10 (i.e. 10 terms), then the **sum** for k = 1 to 100 (100 terms) and so on, and each time we'll ask **MAPLE** to evaluate the **sum** as a decimal (or floating point number) using the **evalf** command ... then we'll watch for the partials **sum**s approaching some limiting value.
Here we go:

```
• Digits:=6;
                                   Digits := 6
• t(k):=1/k;
                                   t(k) := 1/k
• evalf(sum(t(k),k=1..10));
                                     2.92897
• evalf(sum(t(k),k=1..100));
                                     5.18738
• evalf(sum(t(k),k=1..1000));
                                     7.48548
• evalf(sum(t(k),k=1..10000));
                                     9.78761
• evalf(sum(t(k),k=1..100000));
                                     12.0901
• evalf(sum(t(k),k=1..1000000));
                                     14.3927
• evalf(sum(t(k),k=1..10000000));
                                     16.6953
• evalf(sum(t(k),k=1..100000000));
```

```
                                        18.9979
• evalf(sum(t(k),k=1..1000000000));
                                        21.3005
• evalf(sum(t(k),k=1..10000000000));
                                        23.6031
• evalf(sum(t(k),k=1..100000000000));
                                        25.9056
• evalf(sum(t(k),k=1..100000000000000));
                                        32.8134
```

S: Hold on! You've just added 100,000,000,000,000 terms and you're getting nowhere. How long do we have to wait?
P: Who knows ... but let's keep going. Maybe this series converges very slowly. Maybe it takes jillions of terms before we see ...
S: Or maybe we won't live that long. Can't you just tell me the answer? I mean, maybe it doesn't even have a sum ... ever think of that? Maybe it just keeps getting bigger and bigger and ...
P: Aah, you've been taking your smart pills again. What we need is some way to tell if it actually has a sum. If so, we'd just keep going. If not, we wouldn't even bother starting. Do you recognize this series?
S: No. Should I?
P: We've run into it before. It's called the HARMONIC SERIES. In fact, I think I said ... in fact I proved, that

$$\lim_{n \to \infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n}}{\ln n} = 1$$

by considering the area under the curve $y = \frac{1}{x}$ from x = 1 to x = n. Remember?

And do you know what that means? I'll tell you. It means that the values of $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ are very much like $\ln$ n, when n is large. In fact, the ratio $\frac{S_n}{\ln n}$ is very nearly "1". In fact ...

S: So let's see **MAPLE** calculate *ln* n ... just to check it out.
P: Good idea. We'll just **evalf** the logs of n for n = 10 then 100 then 1000 and so on.
S: How does **MAPLE** know we're talking about natural logs?
P: Because **MAPLE** has taken this course and knows that ...
S: Let's get goin'.
P: Okay, but watch $\log_e(n)$ and compare with S_n computed above and convince yourself that they get closer:

```
• evalf(log(10));
                                        2.30259
• evalf(log(100));
                                        4.60517
• evalf(log(1000));
                                        6.90776
• evalf(log(10000));
                                        9.21034
• evalf(log(100000));
                                        11.5129
• evalf(log(1000000));
                                        13.8155
• evalf(log(10000000));
                                        16.1181
• evalf(log(100000000));
                                        18.4207
• evalf(log(1000000000));
                                        20.7233
• evalf(log(10000000000));
                                        23.0259
• evalf(log(100000000000));
                                        25.3284
• evalf(log(100000000000000));
                                        32.2362
```

LECTURE 5

CONVERGENCE of SERIES

A Test for Convergence of an Infinite Series:

Suppose we are given an infinite series of positive terms: $a_1 + a_2 + a_3 + ...+ a_n + ...$ (where we call the n^{th} term a_n) and suppose we test it, to see if it's a geometric series. We'd take the ratio $\frac{a_2}{a_1}$ and $\frac{a_3}{a_2}$ and so on, to see if they were all the same (since that'd make it a geometric series). Suppose they were NOT the same (so it's NOT a geometric series and we're unlikely to have any formula for the partial sums!) BUT we notice that each ratio is less than the number $\frac{1}{2}$. That is, $\frac{a_2}{a_1} < \frac{1}{2}$ and $\frac{a_3}{a_2} < \frac{1}{2}$ and, in general, $\frac{a_{n+1}}{a_n} < \frac{1}{2}$ for n = 1, 2, 3, and so on. That means $\mathbf{a_2 < \frac{1}{2} a_1}$. Further, $a_3 < \frac{1}{2} a_2$ and that means $\mathbf{a_3 < \left(\frac{1}{2}\right)^2 a_1}$. Also, $a_4 < \frac{1}{2} a_3$ means that $\mathbf{a_4 < \left(\frac{1}{2}\right)^3 a_1}$ and, in general, $a_{n+1} < \frac{1}{2} a_n$ means that $\mathbf{a_{n+1} < \left(\frac{1}{2}\right)^n a_1}$. Hence, the given series, namely $a_1 + a_2 + a_3 + ...$ is less, term-for-term, than the series $\mathbf{a_1 + \left(\frac{1}{2}\right) a_1 + \left(\frac{1}{2}\right)^2 a_1 + \left(\frac{1}{2}\right)^3 a_1} + ...$ which (surprise!) is a geometric series with first term a_1 and common ratio $\frac{1}{2}$. Hence, the partial sums of the given series, $a_1 + a_2 + a_3 + ...$ can't possibly become infinite (as they did for the HARMONIC SERIES), because they are always less than the partial sums for the geometric series $a_1 + \left(\frac{1}{2}\right) a_1 + \left(\frac{1}{2}\right)^2 a_1 + \left(\frac{1}{2}\right)^3 a_1 + ...$ and the partial sums of this series are less than the sum of the *infinite* geometric series which is given by: $\frac{a}{1 - r} = \frac{a_1}{1 - \frac{1}{2}} = 2 a_1$. Hence the given series would converge to some limit.

S: Hold on! Are you saying that just because the partial sums for $a_1 + a_2 + a_3 + ...$ can't get bigger than $2a_1$, then they automatically have a limit? I mean, that sounds like hand-waving to me. I mean ...

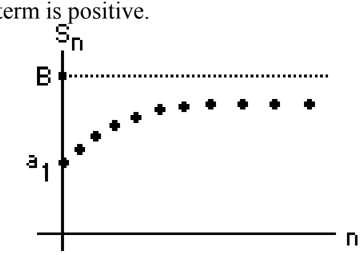
P: Yes, that's what I'm saying. Let's consider the partial sums of some series like $a_1 + a_2 + a_3 + ...$, so $S_1 = a_1$ and $S_2 = a_1 + a_2$ and $S_3 = a_1 + a_2 + a_3$ and so on. We'll plot a graph of S_n versus n and note two things about the graph: first, the sums are increasing, because we keep adding positive terms ...

S: Who said they were positive?

P: Oh, did I forget to mention it? For now we'll only consider series where every term is positive.

S: *Now* he tells me.

P: Okay, first we notice that S_n is an increasing function of n and we also notice that S_n is never larger than $2 a_1$. The graph might look like this ===>>>



In the graph I've assumed that the partial sums, S_n, are never larger than some number B. I don't want to use $2a_1$ because you'll think that it always turns out to be twice the first term!

S: Not me!

P: Okay, here's the theorem: if a sequence is increasing but never gets larger than some number (call it B, for "upper **B**ound") then the sequence converges. That's it! Nice theorem, eh?

S: Are you going to prove it?

P: No, but you must admit ... just by looking at the graph ... that it seems a reasonable theorem. After all, if the graph keeps increasing but can't get larger than B then it must level out and approach some limiting value. See? The theorem just validates common sense thinking. Anyway, we're onto a test for convergence of an infinite series so we shouldn't get sidetracked. But first I'll make a fuss about this new theorem:

> If the sequence $\{S_n\}$ is increasing, but has an upper bound (that is, $S_n \le B$ for some number B),
>
> then $\lim_{n\to\infty} S_n$ exists
>
> ... that is, the sequence *converges to a limit*.

Note that we use the notation $\{S_n\}$ to denote the sequence of numbers: $S_1, S_2, S_3, \ldots$

It's better than referring to "the sequence S_n" because S_n really refers to the n^{th} member of the sequence.

There's another analogous theorem:

> If the sequence $\{S_n\}$ is decreasing, but has a lower bound (that is, $S_n \ge C$ for some number C),
>
> then $\lim_{n\to\infty} S_n$ exists
>
> ... that is, the sequence converges to a limit.

Example: Show that the sequence $\left\{\frac{n}{n+1}\right\}$ converges as $n\to\infty$.

Solution: We want to show: (1) the sequence is increasing, and (2) it's bounded above. Let $a_n = \frac{n}{n+1}$ and consider $\frac{a_{n+1}}{a_n} = \frac{(n+1)/(n+2)}{n/(n+1)} = \frac{(n+1)^2}{n\,(n+2)} = \frac{n^2+2n+1}{n^2+2n} > 1$ so $a_{n+1} > a_n$ hence the sequence is increasing.

However, $a_n = \frac{n}{n+1} < \frac{n+1}{n+1} = 1$ so the sequence is bounded above by the number "1". We conclude that the sequence converges.

S: That's the stupidest thing I ever heard of. I mean, I'd just take the limit: $\lim_{n\to\infty} \frac{n}{n+1} = 1$. That'd mean it converges, right?

P: Sure, but I wanted to demonstrate this new theorem. But let me do one where you can't just find the limit (else, as you say, you certainly wouldn't use these theorems ... you'd just take the limit).

Example: Show that the sequence $\left\{\frac{10^n}{n!}\right\}$ converges.

Solution: Let $a_n = \frac{10^n}{n!}$ and consider $\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10}{n+1} < \frac{10}{11}$ provided $n > 10$. Hence, past the 10^{th} term, the sequence is decreasing. To use the second theorem (above) we need only find a lower bound. But every term is positive so $a_n > 0$. Hence the sequence $\{a_n\}$ is decreasing *and* bounded below by 0 ... hence it converges.

We should get back to what we were saying earlier: if the ratio of successive terms of a series is less than those of a convergent geometric series (where the common ratio satisfies $|r| < 1$), then the infinite series will converge (and, in fact, it will converge to a number less than the sum of the infinite geometric series!).

Example: Show that the series $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges.

Solution: If we call the terms a_1, a_2, etc., then we have $a_1 = \frac{10}{1!} = 10$, $a_2 = \frac{10^2}{2!} = 50$, $a_3 = \frac{10^3}{3!} = \frac{1000}{6}$ and so on. Consider the ratio $\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10}{n+1} < \frac{10}{11}$ provided $n > 10$. Hence, after the 10^{th} term of the series, the terms are less than the terms of a geometric series with common ratio $\frac{10}{11}$ and since this ratio is less than "1", the geometric series converges ... hence the given series converges. We'll make a fuss about this test:

THE RATIO TEST

The series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ (of positive terms) will

<u>converge</u> if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L < 1$, and will

<u>diverge</u> if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L > 1$

Note that the first part is reasonable: if the ratio has a limit less than 1 then the partial sums are less than those of a convergent geometric series, hence they are bounded above, hence these partial sums <u>do</u> have a limit and that's exactly what we mean by "converges". The second part is trickier: if the ratio has a limit greater than 1 then we *could* conclude that the partial sums are greater than the partial sums of a certain geometric series whose partial sums actually become infinite ... so the series we're considering has partial sums which must also become infinite ... so the series diverges. Although we *could* argue in this manner, it's easier to regard the second part of the ratio test as saying that the terms are getting LARGER hence cannot have a limit of ZERO hence the series fails the n^{th} term test, hence diverges.

S: Wait a minute! You were doing this example, and after the 10^{th} term you're okay, but what about the first 10 terms? You did that twice ... once with a sequence and once with a series.

P: It doesn't matter *what* happens for the first 10 or 10,000 terms. For the sequence, I want to show that the sequence has a limit so I just consider the terms beyond the 10^{th}. These DO have a limit as $n \to \infty$ (using the theorem), so <u>preceding</u> these by a few terms doesn't change that fact. After all, we're interested in $n \to \infty$ so what does it matter when n = 1 or 2 or 3 or even n = 10,000? For the series, I just have to show that the partial sums are bounded above. They surely increase because each term is positive, so I'm left with finding an upper bound. That I do by saying

$$a_1 + a_2 + a_3 + \ldots < a_1 + a_2 + a_3 + \ldots + a_{10} + \left(\frac{10}{11}\right) a_{10} + \left(\frac{10}{11}\right)^2 a_{10} + \left(\frac{10}{11}\right)^3 a_{10} + \left(\frac{10}{11}\right)^4 a_{10} + \ldots$$

where I've shown that every term past the 10^{th} decreases by at least $\frac{10}{11}$ so the series on the right (which is my upper bound) is no greater than $a_1 + a_2 + a_3 + \ldots + a_9 + \frac{a_{10}}{1 - \frac{10}{11}} = a_1 + a_2 + a_3 + \ldots + a_9 + 11\, a_{10}$. See? I sum all the terms of the geometric series and get an upper bound for *my* series. Nice, eh? And it didn't matter whether I started at the 10^{th} term or the $10{,}000^{th}$ term. See? Not only that, I can now say that $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges to a number less than

$a_1 + a_2 + a_3 + \ldots + a_9 + 11\, a_{10}$ which is $\frac{10^1}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} + \ldots + \frac{10^9}{9!} + 11\,\frac{10^{10}}{10!}$.

S: And how big is that?

P: I leave it as an exercise for ..

S: For the student ... I know, I know. Anyway, I find this very confusing. Sequences, series ... they all look the same to me. In fact I can hardly tell the difference between the last two examples. One has $\frac{10^n}{n!}$ and the other does too. That's confusing. I mean ...

P: A sequence is a bunch of numbers separated by COMMAS: a_1, a_2, a_3, and so on. A series is a bunch of numbers separated by PLUS signs: $a_1 + a_2 + a_3 + ...$ See? You're adding them, not just inspecting them! If the sequence $\{a_n\}$ approaches a limit of, say, 47 (meaning that $\lim_{n\to\infty} a_n = 47$) then we'd say that the sequence CONVERGES. However, the series $a_1+a_2+a_3+ ...$ would NOT converge if $\lim_{n\to\infty} a_n = 47$. In fact, after adding a few million terms each would look very much like the number 47 so your series would look like ... + 47 + 47 + 47 + ... which certainly doesn't converge -- in fact it becomes infinite! See?

S: But if the terms had a limit, any limit like 0.1, your series would still look like ... + 0.1 + 0.1 + 0.1 + ... after a while. Right? And then it would become infinite. Right? Then it would diverge, right?

P: Very good! And that's *just* what happens unless ... unless what?

S: Huh?

P: If $\lim_{n\to\infty} a_n = L$ then the series $a_1 + a_2 + a_3 + ...$ would look like ... + L + L + L ... after a while, so what value must L have in order to avoid getting infinity for the sum?

S: I haven't the foggiest.

P: How about L = 0? Don't you see? Unless $\lim_{n\to\infty} a_n = L = 0$ the series $\sum_{k=1}^{\infty} a_k$ couldn't possibly add to anything but infinity.

S: So if you give me a series and $\lim_{n\to\infty} a_n$ isn't zero, then the series won't converge, right?

P: Very good! Keep eatin' those smart pills. And we should make a fuss about that because it's probably the easiest test to apply:

the n^{th} term test

The series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$ will diverge if $\lim_{n\to\infty} a_n \neq 0$

Example: Test the series $\sum_{k=1}^{\infty} \frac{10^n}{n^3}$ for convergence.

Solution: We check to see if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{10^n}{n^3} = 0$; if not, the series DIVERGES (meaning it doesn't converge). We can use l'Hopital's rule since $\frac{10^n}{n^3}$ has the form $\frac{\infty}{\infty}$ as $n\to\infty$. Differentiating both numerator and denominator with respect to n we get the ratio $\frac{10^n \, ln\, 10}{3\, n^2}$ which still has the form $\frac{\infty}{\infty}$ so we continue and get $\frac{10^n\, (ln\, 3)^2}{6n}$ and one more time gives $\frac{10^n (ln\, 3)^3}{6}$ which has a limit of ∞ as $n\to\infty$. Since $\lim_{n\to\infty} a_n = \infty$ (rather than the required $\lim_{n \to \infty} a_n = 0$), the series diverges.

S: I though you didn't like to differentiate with respect to an integer. I thought you get nervous when ...

P: I changed my mind. I've decided that I'd just think of "n" as being a continuous variable and do the differentiating without changing its name to "x". Good, eh?

S: Not very. Anyway, I have a question you'll have a problem with. You already said that the Harmonic Series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ + ... adds up to infinity and that means it diverges, right? Yet the terms are $a_n = \frac{1}{n}$ so $\lim_{n \to \infty} a_n = 0$ so the series should converge. How do you like them bananas?

P: I never said that $\lim_{n \to \infty} a_n = 0$ will make a series converge. When did I say that? Pay attention. What I said was

$\lim_{n \to \infty} a_n \neq 0$ will make a series <u>diverge</u>

. That's what I said.

S: Aren't they the same? I mean ...

P: If an animal doesn't have four legs, it's NOT a horse. Got it? Now suppose it *does* have four legs. Is it a horse? Is that a test for a horse? Maybe it's a cow or ...

S: Huh?

P: If $\lim_{n \to \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ is NOT convergent. <u>That's</u> the theorem. If $\lim_{n \to \infty} a_n = 0$ that's not a test for anything.

Maybe the series converges (like $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^n}$, a geometric series which converges to "2") or maybe it diverges (like the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges to infinity).

Remember, if $\lim_{n \to \infty} a_n = 0$, <u>anything</u> can happen.

Example: Does the series $\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!} = \frac{1^2 2^1}{1!} + \frac{2^2 2^2}{2!} + \frac{3^2 2^3}{3!} + \ldots$ converge or diverge?

Solution: We could check to see if $\lim_{n \to \infty} a_n = 0$ with $a_n = \frac{n^2 2^n}{n!}$, but we couldn't use l'Hopital's rule because we have no way of differentiating the denominator, n! = (1)(2)(3)...(n).

S: Hah! So there's a case where you <u>can't</u> consider "n" to be like an "x" ... a *continuous variable* as you call it. I mean, what's (x!) if x isn't an integer. I think you're stuck there!

P: Pay attention. I'm going to use the RATIO TEST.

We look to see if the series is less than a convergent geometric series (with common ratio less than "1"), so we consider the ratio $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^{n+1}/(n+1)!}{n^2 2^n/n!} = \left(\frac{n+1}{n}\right)^2 \frac{2}{n+1}$ which has a limiting value of 0 (i.e. $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0$) which is certainly less than "1" so the series does indeed converge to some limit.

S: And what's that limit?

P: I don't know, but I can now start to add the terms and be guaranteed that they'll add up to something.

S: Are you saying that you're just interested in proving that it adds up to something? Somebody gives you the series and you say "yes, it adds up to something" and you're finished with the problem? Is that what mathematicians do? I mean ...

P: Okay, you've got a point. The ratio of terms is $\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \frac{2}{n+1} \le \left(\frac{11}{10}\right)^2 \frac{2}{11} = \frac{11}{50}$ when $n \ge 10$ so, from the 10^{th} term onward, the series is less than $a_{10} + \left(\frac{11}{50}\right) a_{10} + \left(\frac{11}{50}\right)^2 a_{10} + \left(\frac{11}{50}\right)^3 a_{10} + \ldots = \frac{a_{10}}{1 - \frac{11}{50}} = \frac{50}{39}\, a_{10}$ (using $\frac{a}{1-r}$, the sum of an infinite geometric series) so our series is less than:

$a_1 + a_2 + a_3 + \ldots + a_9 + \frac{50}{39} a_{10}$ or $\boxed{\frac{1^2 2^1}{1!} + \frac{2^2 2^2}{2!} + \frac{3^2 2^3}{3!} + \ldots + \frac{9^2 2^9}{9!} + \frac{50}{39}\frac{10^2 2^{10}}{10!}}$.

S: And what's that?

P: It's ... uh, MAPLE says it adds up to about 44.344, to 3 decimal places.

S: How close is that to the right answer?

P: Well, let's see ... the sum of the series is less than 44.344 and greater than the sum of just the first 9 terms which adds up to ... uh, MAPLE says $\frac{1^2 2^1}{1!} + \frac{2^2 2^2}{2!} + \frac{3^2 2^3}{3!} + \ldots + \frac{9^2 2^9}{9!} = 44.298$ (to 3 decimal places). I conclude that

$$44.298 < \sum_{n=1}^{\infty} \frac{n^2 2^n}{n!} < 44.344$$ and that's a pretty good estimate, eh?

S: Yeah, pretty good, but suppose I wanted the answer to more decimal places ... maybe 6 decimal places. What then?

P: See what we did? We said the sum was <u>greater</u> than the sum of the first 9 terms and <u>less</u> than this sum PLUS the sum of a certain geometric series. If we wanted greater accuracy we'd just keep going past 9 terms to maybe 20 or 30 or 100 terms ... until the geometric series changed the sum by less than, say 10^{-6}.

S: Do it.

P: <u>You</u> do it!

S: Uh ... well ... I'd look at that ratio again: $\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \frac{2}{n+1}$ and if I choose a really big "n" then it'd be pretty small, less than some number "r" and ... let's see ... how small do I want r to be? That extra geometric series would look like $a_n + r\, a_n + r^2 a_n + \ldots$ which adds up to $\frac{a_n}{1 - r}$ and I'd want that to be less than 10^{-6} ... I guess. How'm I doing boss?

P: Keep going, you're doing fine.

S: Okay, I'd want ... uh, I'd want ... I think this is too tough, don't you? I should try a bunch of n's and see if I'm there yet. I mean, less than 10^{-6}.

P: That's a good idea. The ratio $\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \frac{2}{n+1}$ approaches zero as $n \to \infty$, so we can make it less than a really *really* small "r" so when you want $\frac{a_n}{1 - r}$ to be less than 10^{-6} it's something like asking a_n itself to be less than 10^{-6} because, after all, $\frac{a_n}{1 - r}$ is pretty close to a_n if r is very *very* small. Okay, how big must n be so that $a_n = \frac{n^2 2^n}{n!} < 10^{-6}$ or maybe we should ask $\frac{n^2 2^n}{n!} < 10^{-7}$ just to be on the safe side. How big?

S: I haven't the fogg..

P: Let's take logs. We'd be asking that $\log\left(\frac{n^2 2^n}{n!}\right) < \log(10^{-7}) = -7 \log(10)$ and this time we'll actually pick logs to the base 10. Nice, eh? Then we'd want $\log(n^2) + \log(2^n) - \log(n!) < -7$ (since $\log_{10}(10) = 1$). We write $\log(n!) = \log\big((1)(2)(3)...(n)\big) = \log(1) + \log(2) + \ldots + \log(n)$ and that'd mean we need $2 \log(n) + n \log(2) - (\log(2) + \ldots + \log(n)) < -7$ or $\boxed{7 + n \log(2) + \log(n) < \log(2) + \log(3) + \ldots + \log(n-1)}$. Now we look at a table of common logs (to the base 10) and keep adding the logs of the integers until their sum exceeds $7 + n \log(2) + \log(n)$. See? That'd give you a value of "n" and the sum of the series would lie between the sum of the first n terms and <u>this</u> sum PLUS that extra geometric series.

S: I'm sorry I asked ... let's forget the whole thing. I know <u>that</u> won't be on the final exam.

P: Well, at least you've learned something. You've learned that you can get estimates of the sum of an infinite series (provided it converges) and you've even got a couple of tests so you can determine if it <u>does</u> converge and ...

S: Yeah, I've learned something ... but I don't know what good it'll do me, except to pass a final exam. I mean, does this stuff have any *useful* applications ... outside of mathematics?

P: Patience.

LECTURE 6

ALTERNATING SERIES and ABSOLUTE CONVERGENCE

ALTERNATING SERIES

We saw that the HARMONIC SERIES $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges to infinity. We proved this by comparing the partial sums $S_1 = 1$, $S_2 = 1 + \frac{1}{2}$, $S_3 = 1 + \frac{1}{2} + \frac{1}{3}$, etc. with the area under the graph of $y = \frac{1}{x}$ from $x = 1$ to $x = n$, namely $\int_1^n \frac{dx}{x} = \ln n \to \infty$ as $n \to \infty$ and using this we showed that S_n increased without limit ... so the series diverged. Remember! In order for a series to converge, the partial sums must have a limiting value. (That's the definition of "convergence".) We could also have tried the nth term test: If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

This, too, will guarantee that the partial sums will NOT have a limit. However, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ so this test fails to give any information about the convergence or divergence of the harmonic series.

S: But we already know it diverges, so why are you still applying tests?

P: I want to demonstrate that the n^{th} term test and the ratio test sometimes give no information at all about the series. I also want to remind you that the crucial point is that the partial sums must have a limiting value as $n \to \infty$, else the series diverges. Pay attention and I'll get to the point.

Had we tried the RATIO test we'd consider the ratio of successive terms: $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$ and we'd get

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ which is NOT less than 1, hence the partial sums S_n, although increasing (since the terms of the harmonic series are all positive), are NOT bounded above by a convergent geometric series, so the RATIO test also gives no information. **Remember this!**

the n^{th} term test and the RATIO test can fail to establish the convergence or divergence of an infinite series

and **remember this!**

In order to converge, the partial sums of an infinite series must have a limiting value

Okay, I just wanted to make these points before we went on.

S: Just wait a minute! You goofed! For the harmonic series, the ratio $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$ is less than 1, so the RATIO test says the harmonic series converges, right?

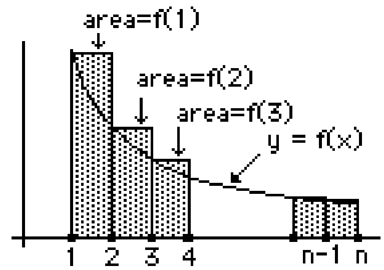
P: Wrong! It is the limiting value of the ratio which must be less than 1 and the limiting value in this case is NOT less than 1. In fact, now that you've mentioned it, it's a good example which shows that even if the terms get smaller (meaning $\frac{a_{n+1}}{a_n} < 1$) the series might *still* diverge. The critical thing is that the terms must get smaller fast! For the harmonic series, they don't get small enough fast enough. After all, the 100^{th} term is only 1% smaller than the 99^{th} term and the 1000^{th} term is only

0.1% smaller, so the terms decrease in size *very* slowly. On the other hand, for the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ the 100^{th} term is 50% of the 99^{th} term and the terms decrease *very* rapidly and after a while they're microscopic in size. Anyway, I wanted to consider a different kind of series where just getting smaller is (almost) enough to guarantee convergence.

S: Wait a minute. That "small enough fast enough" sounds familiar. Haven't we done that before?

P: Yes, you've got a good memory. When we talked about "improper integrals" of the form $\int_a^\infty f(x)\,dx$ we said that in order for the integral to "converge" (see? it's the same word!) the function f(x) has to get *small enough fast enough*. In fact, an improper integral like that is very much like an infinite series. Remember, $\int_a^\infty f(x)\,dx$ is really a $\int$um of terms ... a Riemann SUM ... much like an infinite series.

In fact, if we use the notation f(1) + f(2) + f(3) + ... for our infinite series, rather than $a_1 + a_2 + a_3 + \ldots$ (where f(n) gives the value of the n^{th} term of the series), then we can associate the partial sums $S_n = f(1) + f(2) + \ldots + f(n) = \sum_{i=1}^{n} f(i)$ with an area, as shown ===>>

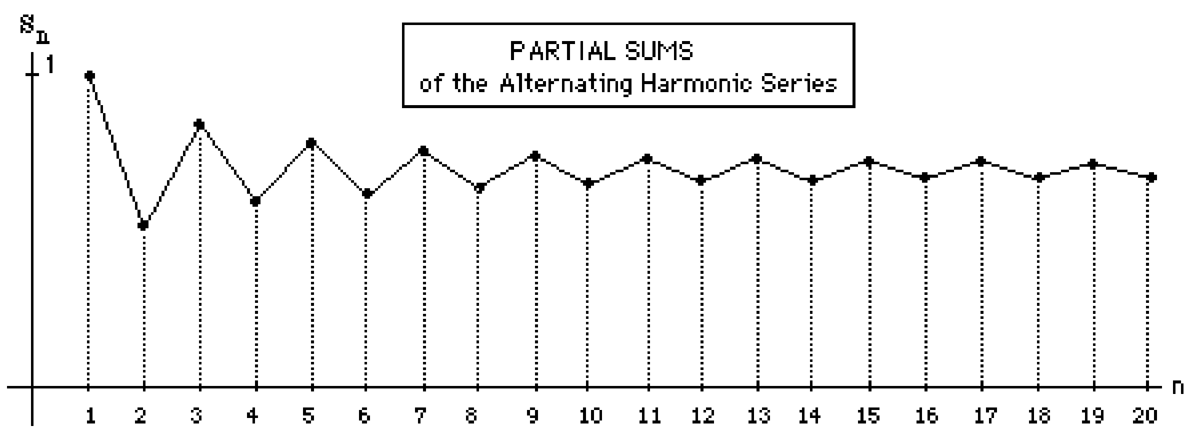


S: Didn't we do something like that when we talked about ... uh, what was it?

P: The HARMONIC SERIES. Yes, we did, but now I want to talk about a related series where it's (almost) enough for the terms to get small. They don't have to get small fast. Pay attention.

the Alternating Harmonic Series

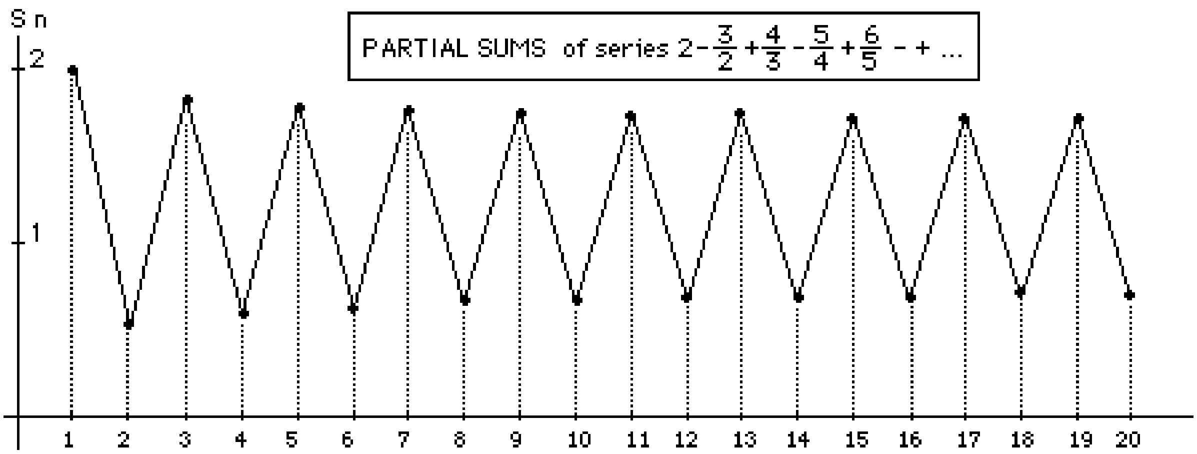
Consider the infinite ALTERNATING HARMONIC SERIES: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \ldots$ where the terms alternate in sign. The partial sums are $S_1 = 1$, $S_2 = 1 - \frac{1}{2}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{3}$, etc. Let's plot these partial sums to see if there is any chance that they approach a limit as $n \to \infty$.



We compute and plot S_1, S_2, S_3, S_4, etc. and note that the partial sums S_n oscillate, the amplitude of the oscillations becoming less and less. In fact, if we call the terms a_1, a_2, a_3 , etc., (meaning the *magnitude* or *absolute value* of the terms, so $a_2 = 1/2$, not $- 1/2$), then starting with $S_1 = a_1$, we subtract a_2 (which is less than a_1) then add a_3 (which is less than a_2) then subtract a_4 (which is less than a_3) and so on. In other words, when we go down (by subtracting) then we go up less than we went down. When we go up (by adding a term) we then go down less than we went up. This is characteristic of <u>any</u> alternating series if the terms get smaller in magnitude ... and for such a series the partial sums oscillate and often converge to some limiting value. In the case of the alternating harmonic series, the partial sums (hence the infinite series) <u>do</u> converge to a number which is roughly .7 (as indicated on the diagram).

S: They "often" converge? Is that what you said? "Often"? How often?
P: Patience.

Consider the series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + - ...$ which is certainly "alternating" (since the terms alternate in sign) and the terms do get smaller in magnitude (or absolute value) since $\frac{2}{1} > \frac{3}{2} > \frac{4}{3} > \frac{5}{4} > \frac{6}{5}$ and so on. Yet this series doesn't converge. To see that, let's look at the partial sums, graphed:



The partial sums don't have a limiting value. In fact, the terms of the series approach 1 in magnitude (even though they're getting smaller) so, after a while, the series begins to look like .... + 1 - 1 + 1 - 1 + - ... and just oscillates without converging to a limit ... hence the series diverges. Clearly we need something more than simply "an alternating series where the terms get smaller". We also need ...

S: I know! The terms ... uh, the terms ... uh ... I thought I had it.
P: The terms must decrease to zero! And that is the test for convergence of an alternating series:

the ALTERNATING SERIES TEST

The alternating series $\sum_{k=1}^{\infty}(-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + - ...$

converges if the terms a_n **decrease to zero**.

Note several things:

- In writing $a_1 - a_2 + a_3 - a_4 + - ...$ we assume that the a's are positive and the alternating sign is shown explicitly in front of each term. (If $a_1 = 1$ and $a_2 = -1/2$ and $a_3 = 1/3$ etc. the series would NOT be alternating, but would, in fact, be the harmonic series ... so when we write $a_1 - a_2 + a_3 - a_4 + - ...$ we're assuming each of a_1, a_2, a_3, etc. are positive.)
- The phrase "**decrease to zero"** means two things:

  $a_{n+1} \le a_n$ (the terms decrease) and $\lim_{n\to\infty} a_n = 0$ and they decrease to **zero**.

- This test is perhaps the easiest test to apply ... so you have to hope that you never run across series other than "alternating" ones!

S: What if the test fails? I mean, your *other* tests failed ... sometimes: the n^{th} term test and the RATIO test.

P: Well, if $\lim_{n\to\infty} a_n \ne 0$, you know the answer, don't you?

S: Huh?

P: That's the n^{th} term test: if $\lim_{n \to \infty} a_n \neq 0$ then the infinite series definitely diverges.

S: You mean that test holds for alternating series too?

P: Yes, it holds for <u>every</u> infinite series, not just ones with positive terms or ones where the terms alternate in sign. For example, consider the series $\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + + - + + - \ldots$ where there are two positive terms then a negative term and so on. It's not alternating and the terms certainly aren't all positive, yet it diverges. Why?

S: I haven't the fogg ... uh, wait ... the terms don't have a zero limit. Right?

P: Right. The limiting value of the terms, namely $|a_n| = \frac{n}{n+1}$, is 1 not 0. Hence we conclude that the series diverges.

S: Will I have to deal with series like that?

P: I'll only expect you to test series for which you actually <u>have</u> a test ... and that says it all.

S: Hmmph.

P: Let me tell you something else about alternating series ... ones in which the terms decrease to zero (so you know the series will converge). This is really nice so pay attention:

Estimating the Sum of a Convergent Alternating Series

If we look again at the graph of partial sums of the convergent alternating harmonic series we note some interesting things:

- The limiting value of the partial sums (hence the sum of the infinite series) is less than S_1 and greater than S_2
- In fact, the partial sums are alternately greater than, then less than, then greater than ... the sum of the infinite series. That means that we can stop adding & subtracting terms and the sum of the infinite series will automatically lie between the last two partial sums we computed.

We make a fuss about this:

For $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + - \ldots$, if the terms a_n decrease to zero, then

$$a_1 - a_2 + a_3 - a_4 + - \ldots - a_{2n} \leq \sum_{k=1}^{\infty} (-1)^{k+1} a_k \leq a_1 - a_2 + a_3 - a_4 + - \ldots - a_{2n} + a_{2n+1}$$

In words:

The sum of an infinite alternating series (whose terms decrease to zero) lies between successive partial sums

(Note that the alternating series must have terms which **decrease to zero**.)

Example:
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \leq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \leq 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \quad \text{and}$$

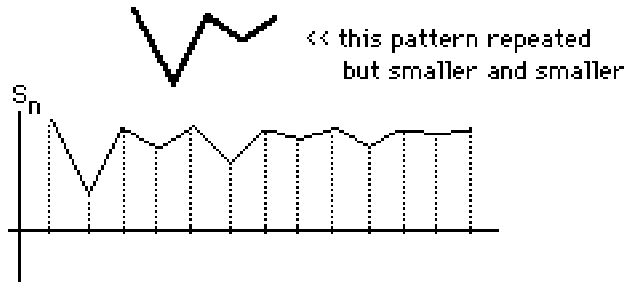
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \leq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \leq 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \quad \text{and so on.}$$

S: I don't get it. Why do you need this decrease to zero stuff? Besides, you didn't answer my last question ... not all of it. I asked what if the test fails ... I mean the alternating series test. What if the terms <u>don't</u> get smaller? Oops, wait, I think I know. If they don't have a limit of zero then the series diverges ... so they <u>have</u> to have a limit of zero, in which case they must get smaller too, right?

P: Wrong! It's possible to have an alternating series where $a_n \to 0$ yet the terms aren't continually decreasing.

S: What! That's impossible ... isn't it? I mean, if they don't decrease, how do they get to zero?

P: You can recognize the plot of partial sums of an alternating series by the fact that they oscillate. You can also recognize that the series converges by the fact that the partial sums have a limiting value as $n \to \infty$. Now, consider the following plot: it's clearly that of a convergent alternating series, and the terms have a limit of zero (since the oscillations die out), YET the terms are NOT decreasing. In fact, I got the plot simply by repeating the Vv pattern, but decreasing the size



somewhat each time. Note that the first dip gets us to $S_2 = a_1 - a_2$ (i.e subtracting a_2 from a_1) then we add a_3, subtract a_4, add a_5 then comes a big subtraction, namely a_6, to begin the Vv pattern again. Clearly $a_6 > a_5$ so the terms are NOT decreasing.

S: Whoa! I don't think such a series exists, do you? Besides, sometimes the terms get smaller and sometimes they don't, right?

P: Right, but the alternating series test is NOT satisfied because the test requires that the terms form a DECREASING SEQUENCE of numbers: $a_1 \geq a_2 \geq a_3 \geq a_4 \geq$ etc. etc. and that's not the case here, yet the series converges. As for the series existing, it should be pretty easy to invent such a series. Let's try $1 - \frac{1}{2} + \frac{1}{2}$ which gives us the first V, then we continue with $-\frac{1}{8} + \frac{1}{8}$ which gives us the second, smaller v, then we continue with all terms decreased by a factor $\frac{1}{2}$ and do the Vv bit again, then decrease by $\frac{1}{2}$ again, and so on. Our series would look like: $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{8} + \frac{1}{8} - \frac{1}{4} + \frac{1}{4} - \frac{1}{16} + \frac{1}{16}$ and so on. See? That's the series, it converges but the alternating series test doesn't apply because the terms don't continually decrease.

S: I think the alternating series test is a lousy test.

P: It's not so bad. **Remember this!** A series will converge or diverge depending upon what EVENTUALLY happens to the terms. If the first few thousand terms don't satisfy any test, but thereafter the terms DO satisfy some test for convergence, then the series will converge. Let's write this out big:

> A series $\sum_{n=1}^{\infty} A_n = A_1 + A_2 + A_3 + \ldots$ will converge (or diverge) if the terms **eventually** satisfy some test for convergence (or divergence).

For example, if a series starts off not being alternating but becomes alternating after the first million terms, and if the terms thereafter **decrease to zero**, then the series will converge.

S: That seems pretty fishy to me. I mean ...

P: Okay, pick a convergent series.

S: Who? Me? Uh ... I pick $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

P: Okay, now I'll place a million terms in front of your series, say 1 + 2 + 3 + ... + 1,000,000. Do you think I can make the resultant series diverge? Certainly not! My terms add up to something huge, but it makes no difference. Eventually we get to your series and they determine whether the *infinite* series converges. I've just added a constant to the sum of your series. I can't make it diverge. If you wanted to prove convergence, you'd just march past my terms and use the ratio test on your series. See?

S: I guess.

Example: Test the following series for convergence.

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ (b) $\sum_{k=1}^{\infty} \frac{1}{k!}$ (c) $\sum_{i=1}^{\infty} (-1)^i \frac{3^i}{i}$

Solution:

(a) It's an alternating series so we apply the alternating series test:

First note that: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0$. Further, we must show that the terms decrease,

either by showing that $\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} \le 1$ or by showing that $a_n - a_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} \ge 0$. We'll do the

latter. Bringing to a common denominator and simplifying gives $a_n - a_{n+1} = \frac{2n - 1}{(n^2+1)((n+1)^2+1)} > 0$ for $n \ge 1$.

Hence the alternating series converges.

(b) We'll use the ratio test (in fact we ALWAYS use the ratio test if factorials are involved): $\frac{a_{k+1}}{a_k} = \frac{k!}{(k+1)!} = \frac{1}{k+1} \to 0$ as $k\to\infty$ and since the limiting value of the ratio is less than 1, the series converges. (In fact, it converges to the number e.)

(c) $\sum_{i=1}^{\infty} (-1)^i \frac{3^i}{i}$ is an alternating series $a_1 - a_2 + a_3 - + \ldots$ with $a_i = \frac{3^i}{i}$. We'll first see if the terms decrease by considering $\frac{a_{i+1}}{a_i}$, to see if it's less than 1. (Last time we considered $a_i - a_{i+1}$ to see if it was positive). This ratio is $\frac{\frac{3^{i+1}}{i+1}}{\frac{3^i}{i}} = 3\frac{i}{i+1} \to 3$ as $i\to\infty$ so the terms are eventually <u>increasing</u>! Since $\lim_{i\to\infty} a_i \neq 0$, the series diverges by the n^{th} term test (which should, here, be called the i^{th} term test!) or by the ratio test (since the limiting value of the ratio is greater than 1).

ABSOLUTE CONVERGENCE

We consider one final test for convergence, just in case the series doesn't have *only* positive terms ... or maybe it's not even alternating. We consider the infinite series $A_1 + A_2 + A_3 + \ldots = \sum_{n=1}^{\infty} A_n$ where the terms A_n may be of <u>any</u> sign. We state without proof the following:

If $\sum_{n=1}^{\infty} \lvert A_n \rvert$ converges , then $\sum_{n=1}^{\infty} A_n$ converges.

In fact, replacing every term by its absolute value will generate a series whose partial sums are larger (or, at least just as large). The resultant series then has only positive terms (or at least non-negative) so we can apply the RATIO test. If the series of absolute values converges, so will the original series. In fact, the original series is said to be ABSOLUTELY CONVERGENT .

Example: Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!}$ for convergence.

Solution: We consider the series of absolute values: $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ to which we apply the RATIO test:

$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n} = \frac{3}{n+1}$ –> 0 as n–>∞ and since this limiting value of the RATIO is less than 1, the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges, hence the original series $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!}$ converges as well ... and, indeed, converges absolutely.

S: Whoa! What's all this about "absolutely convergent"? If it's convergent then it's absolutely convergent ... is that it? It's like saying "If I'm certain then I'm absolutely certain", right? "If it's a horse, then it's absolutely a horse". "If ...

P: No, no, no! Here, the word "absolutely" means we're considering a series where every term has been replaced by its "absolute value". Now we can study this modified series, since every term is now positive and we can use the RATIO test, and if we can show that this modified series converges, the original series will, too. In fact, the original series is said to "converge absolutely". Do you see that?

S: Not really.

P: Okay, let's do some examples.

Example: Test $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - + \ldots$ for convergence and/or absolute convergence.

Solution: If every term is replaced by its absolute value we get $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k}$ which converges by the RATIO test (the ratio of successive terms is constant at $\frac{1}{2} < 1$ ... or we can just recognize it as a geometric series with common ratio $\frac{1}{2}$ which is less than 1). Hence the original, alternating series not only converges, it converges absolutely.

Example: Test $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{1+n^2}$ for convergence and note if it is absolutely convergent:

Solution:

For this series the terms satisfy $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{1+n^2} = 0$ hence satisfy the ALTERNATING SERIES TEST, so the series converges.

S: Oh yeah? You forgot to show that the terms are decreasing. You just showed that they have a limit of zero and you said that wasn't enough and you ...

P: Okay, okay. I could show that they decrease in many ways. First I could consider $a_{n+1} - a_n$ and show that this was ≤ 0 (and that'd mean that $a_{n+1} \leq a_n$) or I could consider $\frac{a_{n+1}}{a_n}$ and show that this was ≤ 1 (and that'd also mean that $a_{n+1} \leq a_n$), but this time I'll do it differently. Pay attention. I'll consider the graph of $y = \frac{x}{1+x^2}$. When x = 1 or 2 or 3 and so on, I'd get the terms in the series, $\frac{n}{1+n^2}$. To show that these terms decrease in size I just have to show that the graph is decreasing, so I consider $\frac{dy}{dx} = \frac{(1+x^2)(1) - x\,(2x)}{(1+x^2)^2} = \frac{1 - x^2}{(1+x^2)^2} \le 0$ when $x \ge 1$ (as is the case if x = 1 or 2 or 3 etc.) Nice, eh?

S: But you haven't checked it for absolute convergence, and the problem said "note, which, if any ..."

P: That's because you interrupted, so let me continue.

The series of absolute values is $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ and we could try the n^{th} term test to see if $\lim_{n \to \infty} \frac{n}{1+n^2} \ne 0$.

Unfortunately, this limit is zero, so the test fails to give us any information. We could also try the RATIO test to see if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$, but, unfortunately, $\lim_{n \to \infty} \frac{\frac{n+1}{1+(n+1)^2}}{\frac{n}{1+n^2}} =$

$\lim_{n \to \infty} \frac{n+1}{n}\,\frac{1+n^2}{2+2n+n^2}$ (now divide numerator and denominator by n^3) $= \lim_{n \to \infty} \left((1+\frac{1}{n})\,\frac{\frac{1}{n^2}+1}{\frac{2}{n^2}+\frac{2}{n}+1} \right) = (1)(1) = 1$

so the RATIO test fails as well. In fact, we have NO test to apply to this series to see if it converges, hence we cannot determine whether the original series converges absolutely.

S: Are you telling me that you don't *know* whether it converges absolutely?

P: No, I'm telling you that none of the tests we've considered will work on the series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$.

S: So, you don't know, right?

P: As a matter of fact I *do* know. The series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges and so the original series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{1+n^2}$ converges, but NOT absolutely.

S: I think you're guessing, right?

P: No, I'm not guessing ... the terms in $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$, namely $\frac{n}{1+n^2}$, look very much like $\frac{n}{n^2} = \frac{1}{n}$ when n is very large and that means the series looks very much like the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series and I *know* that *it* diverges, so the series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges as well. In fact, I might as well add that to the collection of tests we've assembled because it's one of the most useful. But first I should explain what I mean by "the terms of this series look very much like the terms in that series".

Suppose we're given a series of <u>positive</u> terms: $a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$ and, for large values of n, the numbers a_n look like the numbers b_n in the sense that $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ (their ratio approaches a limiting value of

1). Then if $\sum_{n=1}^{\infty} b_n$ converges, so will $\sum_{n=1}^{\infty} a_n$ and if $\sum_{n=1}^{\infty} b_n$ diverges, so will $\sum_{n=1}^{\infty} a_n$.

the COMPARISON TEST

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same convergence properties:
if one converges or diverges, so will the other.

Example: Test the following for convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{n+1}{n\,2^n}$ (c) $\displaystyle\sum_{t=1}^{\infty} \sin\frac{1}{n}$

Solutions:

(a) The terms $a_n = \frac{n^2}{n^3+1}$ "look like" $b_n = \frac{n^2}{n^3} = \frac{1}{n}$ (neglecting the 1 in comparison with the n^3) in the sense that

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$ and since the series $\sum_{n=1}^{\infty} b_n = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the given series also diverges.

(b) The terms $a_n = \frac{n+1}{n\,2^n}$ "look like" $b_n = \frac{1}{2^n}$ since $\frac{a_n}{b_n} = \frac{n+1}{n} \to 1$ as $n\to\infty$, and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, the given series converges absolutely.

(c) The terms $a_n = \sin\frac{1}{n}$ "look like" $b_n = \frac{1}{n}$ since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{t\to 0} \frac{\sin t}{t} = 1$ (where we evaluate the limit by putting $t = \frac{1}{n}$), and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the given series diverges as well.

Example: Test the following series for convergence and/or absolute convergence.

(a) $\displaystyle\sum_{k=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+1}}{n!}$ (b) $\displaystyle\sum_{t=1}^{\infty} \frac{\sin t}{t!}$

Solution:

(a) The terms $a_n = \frac{\sqrt{n+1}}{n!}$ satisfy $\lim_{n\to\infty} a_n = 0$ (we'd have to show this ... it's not enough to simply state it!) and they decrease (we'd have to show this too!) so the series converges by the ALTERNATING SERIES TEST. However, even if we were to pause and <u>prove</u> these statements we'd *still* have to consider the series of absolute values $\sum_{k=1}^{\infty} \frac{\sqrt{n+1}}{n!}$ so we should really consider this <u>first</u>, because if <u>it</u> converges, so does the original series (and we'd avoid having to prove the original series satisfies the alternating series test!). Okay, we'll use the RATIO test and get $\frac{a_{n+1}}{a_n} = \frac{\frac{\sqrt{n+2}}{(n+1)!}}{\frac{\sqrt{n+1}}{n!}} = \sqrt{\frac{n+2}{n+1}} \frac{1}{n+1} \to \sqrt{1}\ (0) = 0$ as $n \to \infty$, hence this series converges (since the limit of the ratio is less than 1). We conclude that the original alternating series also converges ... and it converges <u>absolutely</u>.

S: Whew! I find this very confusing. I mean, there are so many tests ... and I wouldn't know which to use, even if I could remember them all ... and ...

P: Okay, let's review them all. That's important because we're going to use all of them in the next lecture. In fact, it's the next topic that <u>really</u> makes series important. First, let's collect the tests:

<u>REVIEW</u>

For the infinite series $a_1 + a_2 + a_3 + a_4 + \ldots$ we construct the sequence of PARTIAL SUMS

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, etc. and

if $\lim_{n\to\infty} S_n = L$, then we say the infinite series "converges to L".

THE RATIO TEST

The series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ (of positive terms) will

<u>converge</u> if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, and will <u>diverge</u> if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$

(If L = 1, the series may or may not converge)

the n^{th} term test

The series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ will <u>diverge</u> if $\lim_{n\to\infty} a_n \neq 0$

(If $\lim_{n\to\infty} a_n = 0$, the series may or may not converge)

the ALTERNATING SERIES TEST

The alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + - \ldots$

<u>converges</u> if the terms a_n **decrease to zero**.

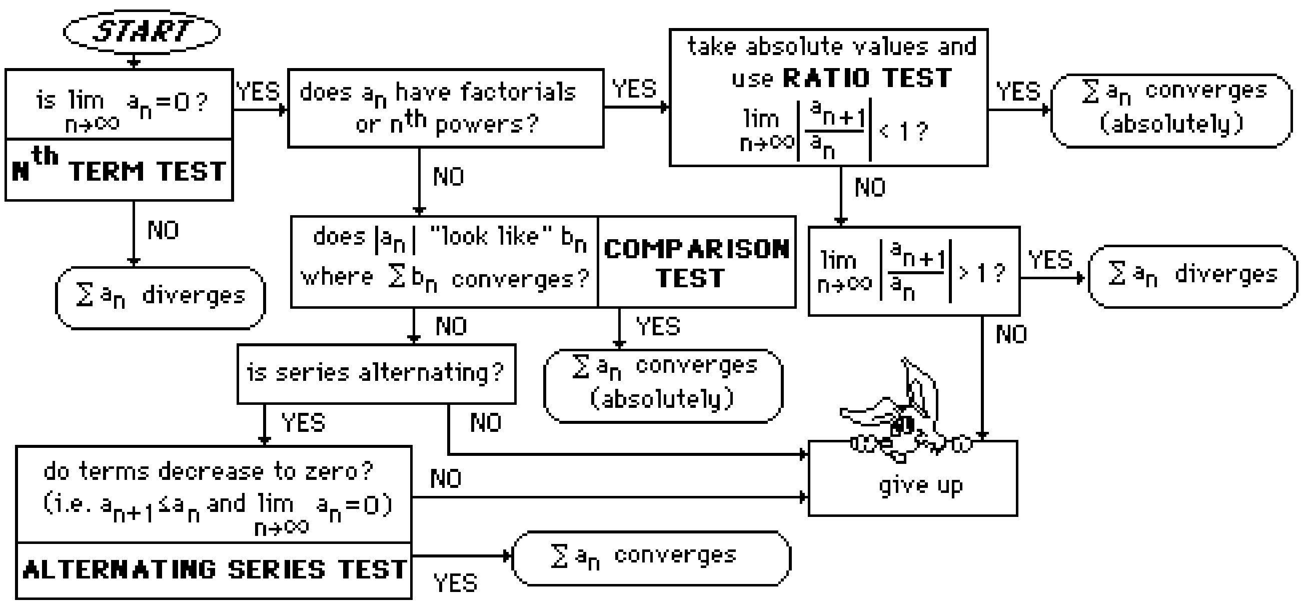
The sum of an infinite alternating series (whose terms decrease to zero) lies between successive partial sums

If $\sum_{n=1}^{\infty} |A_n|$ converges , then $\sum_{n=1}^{\infty} A_n$ converges.

the COMPARISON TEST

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same convergence properties: if one converges or diverges, so will the other.

Given an infinite series, you could use the following scheme:



S: Wheee! That's great ... uh ... is it? I mean, now that I look at it, it seems more confusing than ever. I mean, what if ... uh ... do I have to go in that order, or ...

P: It's just a suggestion, this scheme. Sometimes you recognize a geometric series right away and you needn't even look at the chart. Sometimes you have terms like $\frac{n}{n^2+1}$ which look like $\frac{1}{n}$ which gives the divergent harmonic series ... so again you wouldn't follow the chart. Sometimes you're given $\sum \frac{(-1)^n}{n^2}$ and it's obviously alternating with terms which decrease to zero, so you use the alternating series test directly and don't bother wandering through the chart to find this test. Sometimes ...

S: Yeah, I get the idea. You said all this would come in handy in the next lecture, so can we forge ahead?

LECTURE 7

TAYLOR POLYNOMIALS and TAYLOR SERIES

Recall the technique for finding a polynomial approximation to a given function:
Given, say, $y = f(x) = e^x$, we want a polynomial of degree 5 which approximates this function near x = 0. A polynomial of degree 5 has six constants: $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ and we can impose six conditions on this quintic polynomial so as to get six equations in the six unknowns $a_0, a_1, a_2, ..., a_5$. The conditions we impose are that the polynomial should have the same value as f(x), at x = 0, and the same first derivative and the same second derivative and so on ... ending with the same fifth derivative. (That's six conditions: count them!) This gives:

$$P(0) = f(0),\ P^{(1)}(0) = f^{(1)}(0),\ P^{(2)}(0) = f^{(2)}(0),\ P^{(3)}(0) = f^{(3)}(0),\ P^{(4)}(0) = f^{(4)}(0),\ P^{(5)}(0) = f^{(5)}(0)$$

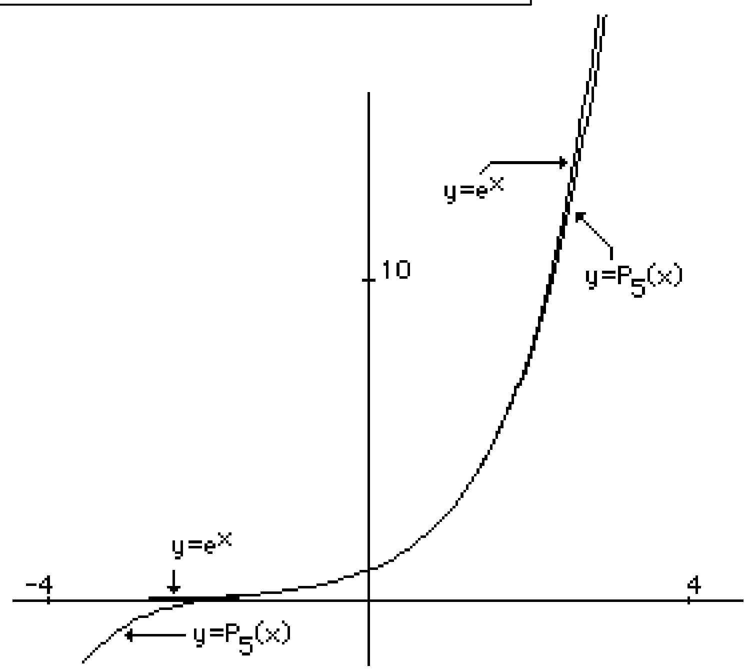
where we use the notation $P^{(3)}$ to mean the 3rd derivative, P'''(x), and so on.
These six equations become:

$$a_0 = e^0 = 1,\ a_1 = e^0 = 1,\ 2\,a_2 = e^0 = 1,\ (3)(2)a_3 = e^0 = 1,\ (4)(3)(2)a_4 = e^0 = 1,\ 5!\,a_5 = e^0 = 1$$

and that means we should choose the coefficients of our polynomial as $a_0 = 1$, $a_2 = \frac{1}{2!}$, $a_3 = \frac{1}{3!}$, $a_4 = \frac{1}{4!}$, $a_5 = \frac{1}{5!}$. The polynomial is then

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

It's easy to check that this polynomial does indeed agree in value and derivatives (up to the 5th) with $f(x) = e^x$.
We plot both y = P(x) and y = f(x) = e^x to see how good the approximation is. In fact, we'll call the 5th degree polynomial approximation $P_5(x)$ since it's clear we could find a polynomial of degree 6 or 7 etc. and we'd expect these to be even better approximations.



The various polynomials which approximate a given function f(x) near, say, x = a, are given by:

TAYLOR POLYNOMIALS

$P_1(x) = f(a) + f'(a)\,(x - a)$ ... which is just the tangent line at x = a

$$P_2(x) = f(a) + f'(a)\,(x - a) + \frac{1}{2!}\,f''(a)\,(x - a)^2$$

$$P_3(x) = f(a) + f'(a)\,(x - a) + \frac{1}{2!}\,f''(a)\,(x - a)^2 + \frac{1}{3!}\,f'''(a)\,(x - a)^3$$

$$P_4(x) = f(a) + f'(a)\,(x - a) + \frac{1}{2!}\,f''(a)\,(x - a)^2 + \frac{1}{3!}\,f'''(a)\,(x - a)^3 + \frac{1}{4!}\,f^{(4)}(a)\,(x - a)^4$$

It's clear how they continue:

- $P_5(x)$ is a polynomial of degree 5 and ends with a term in $(x - a)^5$
- The term which has $(x - a)^k$ also has the kth derivative of f, evaluated at x = a

- This term which has $(x - a)^k$ also has a factor $\frac{1}{k!}$.
- THE THINGS WHICH CHANGE FROM ONE FUNCTION TO ANOTHER ARE THE DERIVATIVES!!!!
- Every Taylor polynomial has the form: **( )** + **( )** (x-a) + **( )** $\frac{1}{2!}$ $(x-a)^2$ + **( )** $\frac{1}{3!}$ $(x-a)^3$ + **( )** $\frac{1}{4!}$ $(x-a)^4$ + ... where you just compute the derivatives of the function f(x), at x = a, and stick them into the appropriate **( )**.

Example: Calculate the Taylor polynomials, about x = 0, for f(x) = sin x.

Solution: We put a = 0 into the general formula (so the powers of (x-a) just become powers of x) and get the scheme:

$$( ) + ( )x + ( )\frac{x^2}{2!} + ( )\frac{x^3}{3!} + ( )\frac{x^4}{4!} + \ldots$$

then we compute the various derivatives of f(x) = sin x at x = 0 and insert them into **( )**. It's perhaps easiest to construct a table of f and its derivatives, and their value at x = a = 0:

n 0	1	2	3	4	5	6	7	etc. etc.	
$f^{(n)}(x)$	sin x	cos x	-sin x	- cos x	sin x	cos x	-sin x -cos x	etc. etc.	
$f^{(n)}(0)$	0	1	0	-1	0	1	0	-1	etc. etc.

Conveniently, the various derivative values just repeat the pattern: 0, 1, 0, -1, 0, 1, 0, -1, etc. etc. so the Taylor polynomials (at x = 0) are particularly easy to construct. $P_7(x)$, for example, is:

$$P_7(x) = \mathbf{(0)} + \mathbf{(1)}x + \mathbf{(0)}\frac{x^2}{2!} + \mathbf{(-1)}\frac{x^3}{3!} + \mathbf{(0)}\frac{x^4}{4!} + \mathbf{(1)}\frac{x^5}{5!} + \mathbf{(0)}\frac{x^6}{6!} + \mathbf{(-1)}\frac{x^7}{7!} +$$

or $P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$. Also, we have $P_1(x) = x$, $P_2(x) = x$, $P_3(x) = x - \frac{x^3}{3!}$, $P_4(x) = x - \frac{x^3}{3!}$,

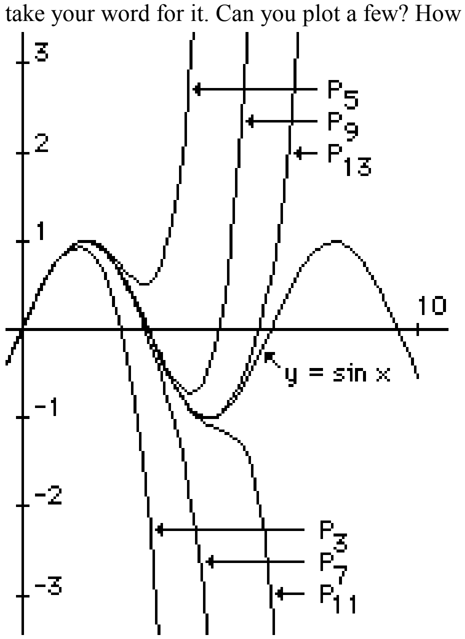
S: Whoa! P_2 isn't even a polynomial of degree two! You said these were polynomials ...

P: Well, that's quite true. We can't call $P_2(x)$ the Taylor polynomial of degree 2 because it's only of degree 1 ... so let's just call it $P_2(x)$, the "second Taylor polynomial", okay? We won't say it's a 2nd degree polynomial. Similarly, $P_4(x)$ isn't a 4th degree polynomial since it's the same as $P_3(x)$ and you'd find that $P_6(x)$ is the same as $P_5(x)$ and $P_8(x) = P_7(x)$ and so on.

S: So what's going on here?

P: If we insist that the second derivative of $P_2(x)$ is the same as the second derivative of f(x) = sin x, at x = 0, then the coefficient of x^2 in $P_2(x)$ will be zero (because the 2nd derivative of sin x is zero, at x = 0) ... so there won't be a term in x^2 ... so $P_2(x)$ will just be 1st degree. In other words, there just isn't a 2nd degree polynomial which has a zero 2nd derivative at x = 0. See?

S: I'll take your word for it. Can you plot a few? How good are they, these P_n approximations?



P: They're quite good so long as you don't stray too far from x = 0. Remember that we're matching derivatives at x = 0. Had we matched at x = 5 then the polynomials would be excellent near x = 5. But just look at those Taylor polynomials trying desperately to match the oscillations of f(x) = sin x. See? The more terms we add the better they do! Notice, too, that the polynomials eventually go off to infinity; they are, after all, polynomials and not sin x which just oscillates between +1 and -1 forever. Can you see which ones go to +∞ and which to -∞?

S: Every other one goes to +∞.

P: Why?

S: I haven't the fogg..

P: When the polynomial ends in a positive term, like $P_5(x)$ ends with $+\frac{x^5}{5!}$, that term takes the polynomial to +∞. When it ends with a negative term, like $P_7(x)$ ends with $-\frac{x^7}{7!}$, then that term takes the poly to - ∞. See? The last term eventually dominates every other term when x is large, so I can tell you right now that

$P_{123}(x)$ will look much like $\pm \frac{x^{123}}{123!}$ when x is very large, the sign depending upon whether the 123rd derivative of sin x is +1 or -1, at x = 0. Near x = 0, of course, $P_{123}(x)$ will look very much like sin x, following the ups and downs for quite a while before taking off to ±∞. See?

Example: Determine the Taylor polynomials, about x = 0, for $f(x) = e^{-x}$.

Solution: We begin by evaluating the various derivatives at x = 0:

n	0	1	2	3	4	5	6	7	etc. etc.
$f^{(n)}(x)$	e^{-x}	$-e^{-x}$	e^{-x}	$-e^{-x}$	e^{-x}	$-e^{-x}$	e^{-x}	$-e^{-x}$	etc. etc.
$f^{(n)}(0)$	1	-1	1	-1	1	-1	1	-1	etc. etc.

Since the polynomial is an approximation about x = **0**, we set a = **0** in the general formula:

$(\ )+(\ )(x-a)+(\ )\frac{1}{2!}(x-a)^2+(\ )\frac{1}{3!}(x-a)^3+(\ )\frac{1}{4!}(x-a)^4+ ...$

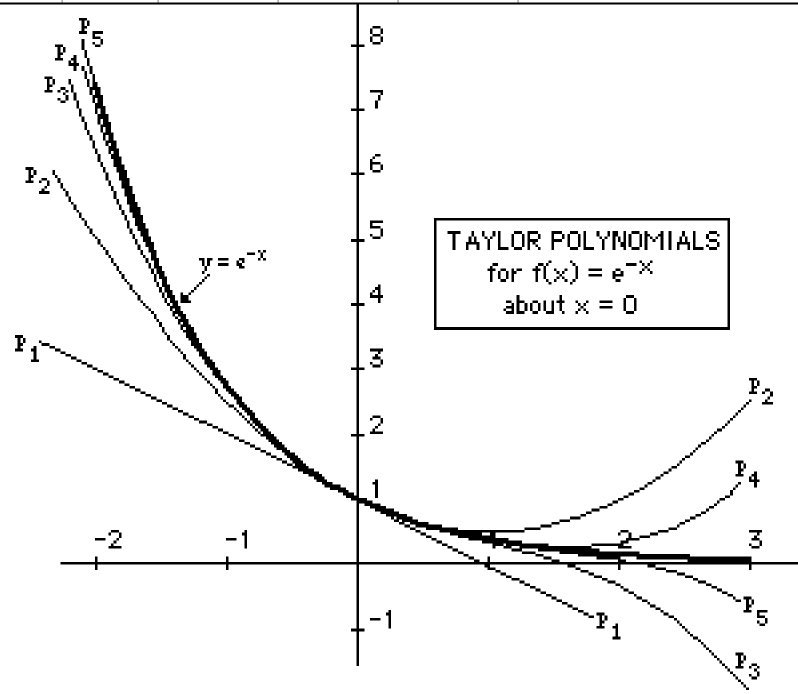
to get

$(\ )+(\ )x+(\ )\frac{x^2}{2!}+(\ )\frac{x^3}{3!}+(\ )\frac{x^4}{4!}+...$

and substitute the derivatives, yielding:

$\mathbf{(1)}+\mathbf{(-1)}x+\mathbf{(1)}\frac{x^2}{2!}+\mathbf{(-1)}\frac{x^3}{3!}+\mathbf{(1)}\frac{x^4}{4!}+ ...$ or simply

$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - + ...$



If we plot the various Taylor polynomials, namely:
$P_1 = 1 - x$,

$P_2 = 1 - x + \frac{x^2}{2!}$, $P_3 = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$, $P_4 = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$ etc. it's clear that they get closer to $y = e^{-x}$, becoming better approximations not only near x = 0 but even farther from this point of approximation. However, $e^{-x} \to 0$ as $x \to \infty$ and that's something a polynomial can't do, so eventually the polynomials leave the exponential.

S: And I notice they take off to either +∞ or -∞ depending on whether the Taylor polynomial ends in a positive or a negative term.

P: Very good! Now here's a curious example. Pay attention and see if you can tell that something different is happening.

Example: Determine the Taylor polynomials for f(x) = *ln* x about x = 1.

Solution: We construct a table of derivatives at x = 1:

n	0	1	2	3	4	5
$f^{(n)}(x)$	*ln* x	$\frac{1}{x}$	$\frac{-1}{x^2}$	$\frac{2}{x^3}$	$\frac{-(2)(3)}{x^4}$	$\frac{(2)(3)(4)}{x^5}$
$f^{(n)}(1)$	0	1	- 1	2!	- 3!	4!

and we can see the pattern of derivatives at x = 1:0, 1, -1!, 2!, -3!, 4!, -5!, etc. etc.

Since the polynomial is an approximation about x = **1**, we set a = **1** in the general formula:

$(\ ) + (\ )(x-a) + (\ )\frac{1}{2!}(x-a)^2 + (\ )\frac{1}{3!}(x-a)^3 + (\ )\frac{1}{4!}(x-a)^4 + ...$ to get

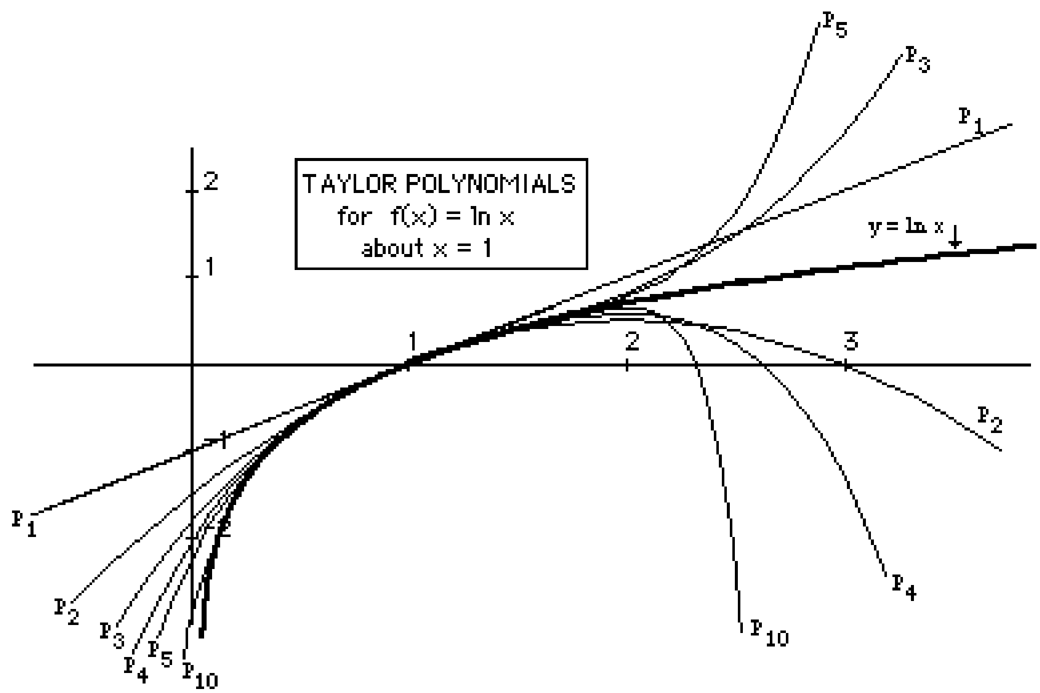
$( ) + ( ) (x\text{-}1) + ( )\frac{1}{2!}\ (x\ \text{-}1)^2 + ( )\frac{1}{3!}\ (x\text{-}1)^3 + ( )\frac{1}{4!}\ (x\text{-}1)^4 + \ldots$ and substitute the derivatives to get

$\mathbf{(0)} + \mathbf{(1)}\ (x\text{-}1) + \mathbf{(-1)}\frac{1}{2!}\ (x\ \text{-}1)^2 + \mathbf{(2!)}\frac{1}{3!}\ (x\text{-}1)^3 + \mathbf{(-3!)}\frac{1}{4!}\ (x\text{-}1)^4 + \ldots$ which simplifies to:

$(x\text{-}1) - \frac{(x\text{-}1)^2}{2} + \frac{(x\text{-}1)^3}{3} - \frac{(x\text{-}1)^4}{4} + \frac{(x\text{-}1)^5}{5} - + \ldots$

Then $P_1(x) = x\text{-}1$ and $P_2(x) = (x\text{-}1) - \frac{(x\text{-}1)^2}{2}$ and $P_3(x) = (x\text{-}1) - \frac{(x\text{-}1)^2}{2} + \frac{(x\text{-}1)^3}{3}$ etc.

We can plot these polynomial approximations and anticipate that they will be excellent near x = 1:



P: Okay, can you see anything different happening?

S: Nope. Looks the same to me. Use a bigger polynomial ... I mean, one of higher degree, and you get a better approx ... wait a minute! P_2 (x) is better than $P_{10}(x)$, or am I seeing things? At least when x is bigger than about 2.5, right?

P: In this example if you want a better and better approximation to *ln* x at, say, x = 2.5, you wouldn't use Taylor polynomials of higher and higher degree. Cute eh? In fact, the higher the degree the worse the approximation.

S: But not at x = .5, right? I mean, it looks like $P_{10}(x)$ is better than $P_2(x)$ if x is close enough to x = 1, right?

P: Right. In fact the higher the degree of the Taylor the better the approximation ... but only for values of x which lie ... where?

S: Uh ... from the picture ... uh, I'd say for x between 0 and 2, maybe more. But how can you tell? I mean, do you know beforehand where the Taylor polys won't be any good? And I've been meaning to ask you ... you seem to like to find the Taylor polys by just getting something like $1 - x + x^2/2 - x^3/3! + x^4/4! - \ldots$ with the DOT DOT DOT at the end, then just picking out the poly you want. But what happens if you keep all the terms. Then you'd get a polynomial that goes on forever. What then?

P: I thought you'd never ask.

LECTURE 8

INFINITE POWER SERIES

TAYLOR SERIES

If we terminate the series $f(a) + f'(a)\ (x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots$ we get a Taylor polynomial for

f(x) about x = a. If we don't terminate, but retain the *infinite* series, we get what is called the **Taylor Series for f(x) about x = a**. It's a special case of a so-called POWER SERIES ... meaning a sum of an infinite number of terms, each of which is a power of (x - a).
Although one might expect that polynomials of higher and higher degree would give better and better approximations ... and the infinite series might give exact values ... that is not always the case.

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the **Taylor Series** for f(x), about x = a

Example: Compute *ln* 1.5 using Taylor polynomial approximations for f(x) = *ln* x about x = 1.
Solution: We use MAPLE to illustrate. First note that the polynomials have the form $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - + \ldots$ and, for x = 1.5, we get $(.5) - \frac{(.5)^2}{2} + \frac{(.5)^3}{3} - + \ldots$ which can be written using Sigma Notation as: $P_n(.5) = \sum_{k=1}^{n} (-1)^{k+1}\frac{(.5)^k}{k}$. We'll use this, telling MAPLE that each polynomial is the **sum** of terms of the form $(-1)^{k+1}\frac{(.5)^k}{k}$, from k = 1 to k = n and we'll write this (for the benefit of the computer) as sum((-1)^(k+1)*.5^k/k,k=1..n)

```
• P1:=sum((-1)^(k+1)*.5^k/k,k=1..1);
                                  P1 := .5
• P2:=sum((-1)^(k+1)*.5^k/k,k=1..2);
                             P2 := .3750000000
• P3:=sum((-1)^(k+1)*.5^k/k,k=1..3);
                             P3 := .4166666667
• P4:=sum((-1)^(k+1)*.5^k/k,k=1..4);
                             P4 := .4010416667
• P5:=sum((-1)^(k+1)*.5^k/k,k=1..5);
                             P5 := .4072916667
• P10:=sum((-1)^(k+1)*.5^k/k,k=1..10);
                             P10 := .4054346479
• P15:=sum((-1)^(k+1)*.5^k/k,k=1..15);
                             P15 := .4054657570
• P20:=sum((-1)^(k+1)*.5^k/k,k=1..20);
                             P20 := .4054650929
• P50:=sum((-1)^(k+1)*.5^k/k,k=1..50);
                             P50 := .4054651084
• log(1.5);
                                .4054651081
```

Notice that the values of $P_n(1.5)$ converge to the exact value: *ln* 1.5 = .4054651081 (to 10 digits). Now we'll do the same, but at x = 2.5

Example: Approximate *ln* 2.5 using Taylor polynomial approximations for f(x) = *ln* x about x = 1.
Solution: We use MAPLE again:

```
• P1:=sum((-1)^(k+1)*1.5^k/k,k=1..1);
                             P1 := 1.5
• P2:=sum((-1)^(k+1)*1.5^k/k,k=1..2);
                         P2 := .375000000
• P3:=sum((-1)^(k+1)*1.5^k/k,k=1..3);
                         P3 := 1.500000000
• P4:=sum((-1)^(k+1)*1.5^k/k,k=1..4);
                         P4 := .234375000
• P5:=sum((-1)^(k+1)*1.5^k/k,k=1..5);
                         P5 := 1.753125000
• P10:=sum((-1)^(k+1)*1.5^k/k,k=1..10);
```

```
                                 P10 := -2.403097097
• P15:=sum((-1)^(k+1)*1.5^k/k,k=1..15);
                                  P15 := 17.95968981
• P20:=sum((-1)^(k+1)*1.5^k/k,k=1..20);
                                 P20 := -96.82858435
• P50:=sum((-1)^(k+1)*1.5^k/k,k=1..50);
                                 P50 := -7590011.528
• log(2.5);
                                     .9162907319
```

Now the approximations become worse as the degree of the Taylor polynomial increases. Indeed, evaluating $P_2(x)$ at x = 2.5 is about the best we can do!

S: That's pretty lousy! I mean, what good are these polys if ...
P: Patience. All will become clear.

CONVERGENCE of TAYLOR SERIES

We look carefully at the infinite series we obtained above for evaluating *ln* 1.5, namely $(.5) - \frac{(.5)^2}{2} + \frac{(.5)^3}{3} - + \ldots$ and test it for convergence using the RATIO TEST. A typical term has absolute value $|a_n| = \frac{(.5)^n}{n}$ so the ratio of absolute values of successive terms is $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(.5)^{n+1}}{n+1}}{\frac{(.5)^n}{n}} = \frac{n}{n+1}(.5) \to .5$ and this is less than 1, so the infinite series converges absolutely. In fact, $(.5) - \frac{(.5)^2}{2} + \frac{(.5)^3}{3} - + \ldots$ converges to *ln* 1.5 = .4054651081 (to 10 digits).

On the other hand, if we consider the series obtained above (presumably for evaluating *ln* 2.5), namely $(1.5) - \frac{(1.5)^2}{2} + \frac{(1.5)^3}{3} - + \ldots$ we get for the ratio $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(1.5)^{n+1}}{n+1}}{\frac{(1.5)^n}{n}} = \frac{n}{n+1}(1.5) \to 1.5$ (as $n \to \infty$) which is greater than 1, so the infinite series diverges (as can be seen from the **MAPLE** calculations). It's not surprising, then, that the Taylor polynomials do NOT provide better and better approximations at x = 2.5, and, in fact, the polynomial values diverge.

S: Hold on ... I've got a great idea. Let's try x = 1.6, 1.7, and so on until we get to 2.5, then we'll see where it fails to converge. Good idea, eh?
P: I have a better idea. Let's just pretend we've substituted one of your values into the Taylor series, then use the RATIO test to see if it converges.
S: But how can you tell ...
P: Patience.

We substitute x = z into $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - + \ldots$ and take the ratio of absolute values to get $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(z-1)^{n+1}}{n+1}}{\frac{(z-1)^n}{n}}\right| = \frac{n}{n+1}|z-1| \to |z-1|$ and we need this to be less than 1 for convergence, that is $|z-1| < 1$ or $-1 < z-1 < 1$ or $0 < z < 2$. Conclusion? The Taylor series $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}$ converges when x = z and $0 < z < 2$ and diverges if $z < 0$ or $z > 2$. Or, to put it more simply:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \quad \text{converges if } 0 < x < 2 \text{ and diverges if } x < 0 \text{ or } x > 2.$$

S: How do you know that? I mean, the bit about divergence?

P: If the limit of the ratio is greater than 1, the series will diverge. See? We substitute x = z and get the limiting value of the ratio to be $| z - 1 |$ ... then we ask: "when is this greater than 1?" It's when $| z - 1 | > 1$ and we have to solve this inequality. Let's see you do it.

S: Huh? Me? Uh ... well, let's see. I want $| z - 1 | > 1$ so that means ... uh ... I give up.

P: I'll give you a hint. Pretend you're asking: "when is $| u | > 1$?" Maybe it's the z - 1 that scares you ... so what values of u will make $| u | > 1$?

S: That's easy. Either $u > 1$ or $u < -1$. Good, eh?

P: Now get back to the problem at hand.

S: Oh, yeah, we want $| z - 1 | > 1$ so that means either $z - 1 > 1$ or $z - 1 < -1$, hence either $z > 2$ or $z < 0$. Hey! That's what *you* got!

P: Pay attention and we'll do some more, but first let me tell you what you can expect:

> A series $a_0 + a_1 (x - a) + a_2 (x - a)^2 + a_3 (x - a)^3 + ... = \sum_{n=1}^{\infty} a_n (x - a)^n$
> will always converge for all values of x in some interval about x = a

Depending upon the coefficients a_0, a_1, a_2, ... any of the following possibilities may occur:

- The series converges *only* for x = a and for no other values of x (so the interval about x = a has zero width).
- The series converges for all values of x in an interval centred on x = a, of the form $a - R < x < a + R$ or perhaps $a - R \le x < a + R$ or perhaps $a - R < x \le a + R$ or perhaps $a - R \le x \le a + R$.
- The series converges for *every* real number x (so the interval about x = a has infinite width).

If the series converges only for values of x lying in some interval such as $a - R < x < a + R$, then the number R is called the RADIUS OF CONVERGENCE of the series. For example, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$ converges in an interval about x = 1, namely $0 < x < 2$, and the *radius of convergence* is R = 1 (since the interval $0 < x < 2$ can be described by $1 - R < x < 1 + R$ with R = 1). We can also check that, when x = 0, the series becomes $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series ... so the series diverges at x = 0. However, the series does, in fact, converge at x = 2. To see this, we substitute x = 2 and get $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ + - ... which is the convergent alternating harmonic series. Finally, when x > 2 or x < 0 the RATIO test tells us that the limiting value of the ratio exceeds 1, so the series definitely diverges. We conclude that the only values of x for which the series converges are those lying in $0 < x \le 2$. The radius of convergence, however, is still R = 1.

S: Hey! If $ln\ x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - + ...$ and if this series converges at x = 2, then it must mean that $ln\ 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - ...$, right? I mean, the alternating harmonic series actually adds up to the number *ln* 2.

P: You got it.

Note: The Taylor series about x = 0 has another name: it's also called the **Maclaurin Series**

Example: The Maclaurin series (or Taylor series about x = 0) for each of the following functions is given. Determine its radius of convergence:

(a) $f(x) = e^x$ has Taylor series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(b) $f(x) = \ln(1 + x)$ has Taylor series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - ... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

(c) $f(x) = \sin x$ has Taylor series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - ... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$

(d) $f(x) = \cos x$ has Taylor series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - ... = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!}$

Solution:

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \frac{1}{n+1} |x| \to 0$ as $n \to \infty$ for every value of x, so the series converges (absolutely) for every value of x. In fact, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ for <u>all</u> x.

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{x^{n+1}}{n+1}\right|}{\left|\frac{x^n}{n}\right|} = \frac{n}{n+1} |x| \to |x| < 1$ when $-1 < x < 1$, and the series converges for these values of x.

In fact, as long as $-1 < x \le 1$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - ...$ (which also gives $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - ...$)

(c) The ratio of successive terms (in absolute value) is $\frac{\frac{|x|^{2n+1}}{(2n+1)!}}{\frac{|x|^{2n-1}}{(2n-1)!}} = \frac{x^2}{(2n+1)(2n)} \to 0$ as $n \to \infty$ so this series converges for every x. In fact, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - ...$ for <u>all</u> x.

(d) The ratio of successive terms (in absolute value) is $\frac{\left|\frac{x^{2n+2}}{(2n+2)!}\right|}{\left|\frac{x^{2n}}{(2n)!}\right|} = \frac{x^2}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$ so this series converges for every x. In fact, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - ...$ for <u>all</u> x.

S: Hold on! For (a) and (b) you talk about $\frac{a_{n+1}}{a_n}$, but for (c) and (d) you just say "the ratio of successive terms". Something fishy goin' on here, right?

P: Well, okay, I just wanted to avoid confusing you. You see, it's not really important, when you write the ratio as $\frac{a_{n+1}}{a_n}$, whether the expressions you substitute for a_{n+1} and a_n are <u>really</u> the $(n+1)^{st}$ and n^{th} terms. They may be the $(n+2)^{nd}$ and $(n+1)^{st}$. See? You need only take the ratio of <u>successive</u> terms, then find the limit as $n\to\infty$. In particular, for the sine series, I really don't want to waste time finding out whether $\frac{x^{2n+1}}{(2n+1)!}$ is the $(n+1)^{st}$ term or the n^{th} term ... I just want to write down two successive terms and take their limit. See? In fact, for the e^x series, namely $1 + x + \frac{x^2}{2!} + ...$ the term $\frac{x^n}{n!}$ is actually the $(n+1)^{st}$ term. See?

S: It isn't really that hard to find out what the n^{th} term is. I can show you how if you'd like. Just count 'em. For the sine series: $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ the 1st term has the x, the 2^{nd} has the x^3, the 3^{rd} has the x^5 ... uh, that's confusing, isn't it? I mean, what's the 100^{th} term, for example?

P: Do you really want to know?

S: Will it be on the final exam?

Example: Prove that the series $\sum_{n=1}^{\infty} \frac{n!}{x^n}$ diverges for every value of x different from 0 (... the terms aren't numbers when x = 0!)

Solution: The ratio of successive terms (in absolute value) is $\frac{\left|\frac{(n+1)!}{x^{n+1}}\right|}{\left|\frac{n!}{x^n}\right|} = (n+1)\frac{1}{|x|} \to \infty$ regardless of the value of x, so the ratio test says the series will diverge no matter *what* x is substituted (for $x \neq 0$, of course).

S: Wait a minute! You said that a series will always converge in some interval about x = a, didn't you?

P: I was talking about a series of the form $a_0 + a_1 (x - a) + a_2 (x - a)^2 + a_3 (x - a)^3 + ... = \sum_{n=0}^{\infty} a_n (x - a)^n$ and the series $\sum_{n=1}^{\infty} \frac{n!}{x^n}$ doesn't have that form. In fact, a series of the form $a_0+a_1(x - a)+a_2 (x - a)^2+a_3 (x - a)^3+...$ is called a **POWER SERIES**. It's pretty obvious that a power series will always converge at x = a. After all, every term is zero except the first ... so it definitely converges. See? The big question is: "How far can x move from 'a' and still give a convergent series?" Remember, the terms get bigger when (x - a) gets bigger, and to converge, the terms have to get small enough fast enough, as n increases. Sometimes even the tiniest deviation from x = a and the series diverges. Sometimes x can move arbitrarily far from x = a and the series will <u>still</u> converge. In that case, the coefficients themselves get so small so fast that even large values of (x - a) won't hinder this convergence.

S: You mean the numbers $a_1, a_2, a_3,...$ can get so small that even if x - a = 1,000,000,0000 the series $a_0+a_1(x - a)+a_2 (x - a)^2+a_3 (x - a)^3+...$ will *still* converge? That's hard to believe.

P: You haven't been paying attention. We've already seen that in earlier examples ... but I'll do another .

Example: Determine the Taylor series for sin x about $x = \frac{\pi}{2}$ and calculate its radius of convergence.

Solution: We construct a table of derivatives, evaluate each at $x = \frac{\pi}{2}$, then substitute into

$$f(a) + f'(a) (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + ...$$ with $a = \frac{\pi}{2}$. That is, we substitute for **( )** in:

$$(\;) + (\;) (x - \tfrac{\pi}{2}) + (\;)\frac{1}{2!} (x - \tfrac{\pi}{2})^2 + (\;)\frac{1}{3!} (x - \tfrac{\pi}{2})^3 + (\;)\frac{1}{4!} (x - \tfrac{\pi}{2})^4 + ...$$

(It's important to note that <u>every</u> Taylor series "about $x = \frac{\pi}{2}$ " will have this form. It's only the derivatives of the function that change from one series to another!)

n	0	1	2	3	4	5	6	7	etc. etc.
$f^{(n)}(x)$	sin x	cos x	- sin x	- cos x	sin x	cos x	-sin x	-cos x	etc. etc.
$f^{(n)}(\frac{\pi}{2})$	1	0	-1	0	1	0	-1	0	etc. etc.

These clearly repeat the pattern: 1 0 -1 0 1 0 -1 0 and so on, so we just substitute and get:

$\mathbf{(1)} + \mathbf{(0)}\,(x - \frac{\pi}{2}) + \mathbf{(-1)}\,\frac{1}{2!}\,(x - \frac{\pi}{2})^2 + \mathbf{(0)}\,\frac{1}{3!}\,(x - \frac{\pi}{2})^3 + \mathbf{(1)}\,\frac{1}{4!}\,(x - \frac{\pi}{2})^4 + \ldots$ or

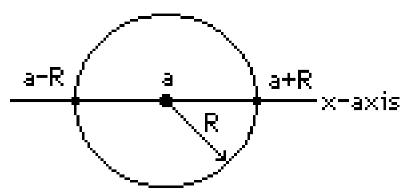
$1 - \frac{1}{2!}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4 - + \ldots$ and if we take the ratio of successive terms (in absolute value) we get:

$$\left| \frac{\dfrac{(x - \frac{\pi}{2})^{2n+2}}{(2n+2)!}}{\dfrac{(x - \frac{\pi}{2})^{2n}}{(2n)!}} \right| = \frac{1}{(2n+2)(2n+1)}(x - \frac{\pi}{2})^2 \to 0 \text{ as } n \to \infty$$

regardless of the value of x; the series converges for <u>all</u> x.

In fact, $\sin x = 1 - \frac{1}{2!}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4 - + \ldots$ for all x. You may recognize this series. Compare it to the series previously obtained for cos x, namely $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots$ so we conclude that $\sin x = \cos(x - \frac{\pi}{2})$ which is a familiar trig identity!

S: Okay, so it converges for all x ... but you asked for the radius of convergence. What <u>is</u> it?
P: It's $R = \infty$. Sounds funny, eh? If a series converges for all x we say it has an infinite radius of convergence.
S: That's not so funny as the name: "radius" of convergence. I don't see any circles around, so how come "radius"?
P: Well, for the general power series $a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \ldots$ which might be some Taylor series about x = a, if the radius of convergence is R, then we imagine x = a as being the centre of a circle of radius R. All values of x inside this circle will yield a convergent series. See?
S: A picture is worth a thousand words. Remember?
P: Okay, here's the picture ===>>>
Happy?
S: Yeah ... happy.

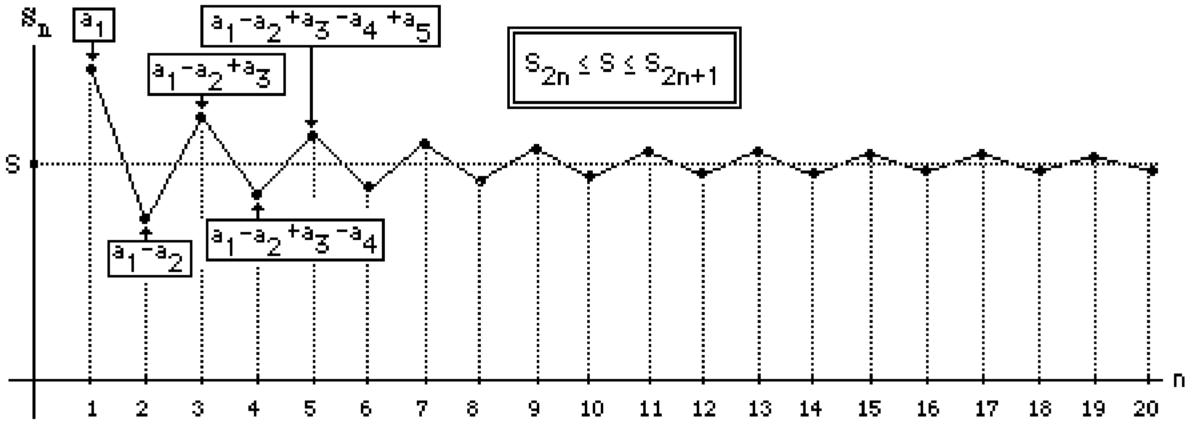


<u>LECTURE 9</u>

MORE ON SERIES

Estimating the Sum of an Alternating Taylor Series

We've already seen that an alternating series $a_1 - a_2 + a_3 - a_4 + - \ldots$ converges if the terms a_n *decrease to zero*. If this is the case then every successive pair of partial sums provides an upper and lower estimate of the infinite sum. By this I mean that, if $S = a_1 - a_2 + a_3 - a_4 + - \ldots$ is the sum of this convergent series, then $S \le a_1$ and $S \ge a_1 - a_2$ and $S \le a_1 - a_2 + a_3$ and so on. To remind ourselves of this we sketch, again, the partial sums of such a series:



Note that the partial sums converge to some limiting value, the SUM of the infinite series ... and we're calling this S. Note, too, that every time we subtract a term the partial sum is *less* than S and every time we add a term the partial sum is *greater* than S. That means we can stop summing terms whenever we get tired and look at the last two partial sums we've obtained. The sum of the *infinite* series lies between these two numbers. In fact, we can restate this by saying **the error is less than the first neglected term**, meaning that the last partial sum we computed is nearer S than the very next term we've neglected. In the plot above, had we stopped summing after 5 terms, then $a_1 - a_2 + a_3 - a_4 + a_5$ is within a_6 of the infinite sum. This is particularly useful when we use a Taylor series to calculate the value of a function --- if we're lucky, the Taylor series will be alternating!

S: Can you really prove that just by looking at the graph? I mean, is that a proof that $S \le a_1$ and $S \ge a_1 - a_2$ and $S \le a_1 - a_2 + a_3$ and so on? What if I had a lousy graph? What if ...

P: Okay, that's a good point. Now watch this ... the magic of brackets. I write $S = a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - - ...$ See? I've just inserted brackets and the numbers inside each bracket are positive (because the terms are getting smaller so $a_2 \ge a_3$ and $a_4 \ge a_5$ etc.) so S is a_1 minus a bunch of stuff. See? That makes $S \le a_1$. In the same way I can write $S = a_1 - a_2 + (a_3 - a_4) + (a_5 - a_6) + + ...$ so S is $a_1 - a_2$ plus a bunch of stuff. See? That makes $S \ge a_1 - a_2$. Nice, eh? And I can do this any number of times and show that S always lies between two successive partial sums. That's the magic of brackets.

Example: Calculate sin 47˚ with an error less than $10^{-5} = .00001$ using an appropriate Taylor series.

Solution: Since the Taylor series for sin x, namely $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + ...$ converges for all x, we can use it to compute sin 47˚. To this end we substitute x = 47˚ in RADIANS (!!!), that is, $x = \left(\frac{\pi}{180}\right) 47$ and get

$$\sin 47^\circ = \left(\frac{47\pi}{180}\right) - \frac{1}{3!}\left(\frac{47\pi}{180}\right)^3 + \frac{1}{5!}\left(\frac{47\pi}{180}\right)^5 - + ...$$ and since the series is alternating, we'll keep summing terms until we reach a term with magnitude less than 10^{-5} ... then we'll stop. This gives:

$$\sin 47^\circ \approx .8203047485 - .5519829673 + .3714292726 - .2499347131 + - ... \quad + .000008405011437$$

where we've had to sum 30 terms until we got to $\frac{1}{59!}\left(\frac{47\pi}{180}\right)^{59} = .000008405011437$, a term with a magnitude less than .00001 and that means ...

S: That means Taylor series are lousy.

P: Hold on. I did this to illustrate a point.

Of course, using the Taylor series about x = 0 is pretty silly since 47˚ is a long way off and although the series will converge at $x = \frac{47\pi}{180} = 47^\circ$ is takes a lot of terms before they start to get really small. It would have been better to use the Taylor series about, say, $x = \frac{\pi}{4} = 45^\circ$ since the various derivatives of sin x (which we'll need to determine the series) are easily computed at 45˚. So we use

$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots$ with $a = \frac{\pi}{4}$. That is, we substitute for **( )** in:

$$(\,) + (\,)(x - \tfrac{\pi}{4}) + (\,)\frac{1}{2!}(x - \tfrac{\pi}{4})^2 + (\,)\frac{1}{3!}(x - \tfrac{\pi}{4})^3 + (\,)\frac{1}{4!}(x - \tfrac{\pi}{4})^4 + \ldots$$

We'll need to compute the derivatives at π/4:

n 0	1	2	3	4	5	6	7	etc. etc.
$f^{(n)}(x)$	sin x	cos x	- sin x	- cos x	sin x	cos x	-sin x -cos x	etc. etc.
$f^{(n)}(\frac{\pi}{4})$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}$	etc. etc.

The pattern is clear, so we just substitute and get:

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \tfrac{\pi}{4}) - \frac{1}{\sqrt{2}}\frac{1}{2!}(x - \tfrac{\pi}{4})^2 - \frac{1}{\sqrt{2}}\frac{1}{3!}(x - \tfrac{\pi}{4})^3 + \frac{1}{\sqrt{2}}\frac{1}{4!}(x - \tfrac{\pi}{4})^4 + \frac{1}{\sqrt{2}}\frac{1}{5!}(x - \tfrac{\pi}{4})^5 - - + + \ldots$$

Now we substitute x = 47° (in RADIANS!!!) or, better, substitute $x - \frac{\pi}{4} = 2°$ (in RADIANS!!!) $= \frac{\pi}{180}(2) = \frac{\pi}{90}$.

Now $\frac{\pi}{90}$ is small, about $\frac{1}{30}$, so the terms will get small quickly and we won't have to sum too many before we reach one that's less than 10^{-5}.

S: Ha! Gotcha! Your series ain't alternating!

P: Uh ... yes, you're quite right ... uh, well, we'll just have to add two terms then subtract two terms then add two and so on ... so we can consider it alternating if we take pairs of terms, right?

S: Huh?

P: If I have a series $a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7 - a_8 + + - -$ ... then I can rewrite it as $(a_1 + a_2) - (a_3 + a_4) + (a_5 + a_6) - (a_7 + a_8) + -$ ... and voila! I have an alternating series! Nice, eh? That's the magic of brackets!

S: Sounds like cheating to me.

P: The only thing I have to watch for is the error. I stop summing my sine series when two terms add to less than 10^{-5}. See? I have an alternating series where each term is the sum of two terms of the original series ... so I keep summing until ...

S: Yeah, yeah, I see.

Substituting $x - \frac{\pi}{4} = \frac{\pi}{90}$ gives the "alternating" series:

$$\left\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{\pi}{90}\right)\right\} - \left\{\frac{1}{\sqrt{2}}\frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{\sqrt{2}}\frac{1}{3!}\left(\frac{\pi}{90}\right)^3\right\} + \left\{\frac{1}{\sqrt{2}}\frac{1}{4!}\left(\frac{\pi}{90}\right)^4 + \frac{1}{\sqrt{2}}\frac{1}{5!}\left(\frac{\pi}{90}\right)^5\right\} - + \ldots$$

= {.7071067814+.02468268300} - {.0004307940867+.000005012516806} + {.0000000437+.0000000003} ...
= 0.7317894644 - 0.0004358066 + 0.0000000440 - +
≈ 0.73135 which is sin 47° with an error less than 10^{-5}.

Note how quickly the terms decrease because of the increasing powers of $\frac{\pi}{90}$ ... and the $\frac{1}{n!}$ helps too, and we only needed 4 terms. We can neglect the terms $\left\{\frac{1}{\sqrt{2}}\frac{1}{4!}\left(\frac{\pi}{90}\right)^4 + \frac{1}{\sqrt{2}}\frac{1}{5!}\left(\frac{\pi}{90}\right)^5\right\}$ which add only 0.0000000440, and that's less than the specified error.

S: I have a theorem for you. Whenever you've got an n! in the denominator, the series will converge. How's that?

P: Not good. Remember that a Taylor series has the form $f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots$ so there's an n! in each denominator ... yet Taylor series don't always converge for all values of x. Sometimes, the derivatives of f(x) get larger even faster than the n! in the denominator, and as (x-a) gets larger, the terms might get too large ... or at least not "small enough fast enough". That poor n! has a lot to do if it wants to drag the terms to zero fast enough. Sometimes it loses the battle. That's the case with *ln* (1+x), remember? If x = 2 (for example), the taylor series about x = 0 diverges. However, n! often succeeds. Look at the series for $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ This series will converge even if x = 1000 or 1,000,000 or 10^{100}. That's because n! wins against powers of x, no matter how large x gets.

S: You mean n! is really big, right? I mean, when n gets large, then n! is much larger than $1,000,000^n$. I'm not sure I really believe that. I mean, $1,000,000^{100}$ is pretty big ... <u>much</u> bigger than 100!
P: Do you want to know how big 100! is?
S: Oh oh ... I can see this coming. Don't tell me. It's huge, right?
P: Much too large for my $4.95 calculator, so I'll ask MAPLE.

```
• evalf(100!);
                                              158
                                 .9332621544*10
```

S: But that's not bigger than $1,000,000^{100}$ ... I mean, $1,000,000^{100} = (10^6)^{100} = 10^{600}$ and that <u>is</u> much bigger, just like I said.
P: What that means is, when you substitute x = 1,000,000 into $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ you have to take a lot of terms before they start to get small enough fast enough; but they will, believe me.
S: I hate to make a fuss about this ... but I *don't* believe you. I mean ...
P: Okay, let's look at $1,000,000^n$ versus n!, as n increases. First, notice that $1,000,000^n$ is a product of n factors, and each one is identical, namely 1,000,000. Now look at n! which is also a product of n factors ...
S: Yeah, but they're (1)(2)(3) ... and that means they're small!
P: Patience ... n! can wait around until its factors are MUCH larger. Remember, $1,000,000^n$ just has factors of 1,000,000 but n! eventually has factors of $1,000,000^{1,000,000}$ and even larger. Do you really think that $1,000,000^n$ can keep up? Not likely. In fact, let's compare $1,000,000^n$ with n! by taking their ratio, After all, that's precisely what we're doing when we investigate the size of terms $a_n = \frac{x^n}{n!}$, for x = 1,000,000. Now these terms eventually <u>decrease</u>, meaning that n! is increasing more rapidly then $1,000,000^n$. If we look again at what the RATIO test is telling us, we consider $\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1}$ which <u>is</u> less than 1 when n > x. In other words, if x = 1,000,000, the terms a_n will start to decrease after the $1,000,000^{th}$ term because n+1 > x. See? Thereafter, the n! really goes to town. After 2,000,000 terms they're decreasing by at least $\frac{x}{n+1} = \frac{1,000,000}{2,000,000} = \frac{1}{2}$ and after 10,000,000 terms they're decreasing by at least $\frac{1,000,000}{10,000,000} = \frac{1}{10}$ and after ...
S: Okay, I get the idea. So n! gets big ... if we wait long enough. But I know something even bigger.
P: What?
S: e^n is bigger, because e^x grows exponentially and you always said that ...
P: You can't be serious? e^n isn't even as large as $1,000,000^n$ because e isn't as large as 1,000,000.
S: But what about all that explosive exponential growth stuff?
P: Remember that $1,000,000^x$ is also an exponential function, just like e^x ... and it grows faster ... but not so fast as n! because ...
S: Okay, tell me something that grows faster than n!
P: $e^{n!}$
S: Aha! And $1,000,000^{n!}$ grows even faster than that!
P: Smart fellow. Now where were we?

In estimating the sum of an infinite series, pray that it's alternating with terms which <u>decrease to zero</u>. Then we can stop summing at any time and the error will be less than the first neglected term. Indeed, the sum of the infinite series will lie between the last two partial sums.

Example: Estimate $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - + ...$

Solution: We add a few terms and get: $S_1 = 1$, $S_2 = 1 - \frac{1}{2} = .5$ (so we now know that the infinite series adds to something between .5 and 1) and $S_3 = 1 - \frac{1}{2} + \frac{1}{4} = .75$ (so we know the infinite series adds to something between .5 and .75) and $S_4 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = 0.625$ (so we now know that the infinite series adds to something between .625 and .75) and ...

S: Hold on! That's a geometric series and I know *exactly* what it adds up to ... it's ... uh, $\frac{a}{1 - r} = \frac{1}{1 - (-\frac{1}{2})} = \frac{2}{3}$.

P: And $\frac{2}{3}$ does indeed lie between .625 and .75, so aren't you impressed?

S: I'd be more impressed if you didn't have to rely on the series being alternating. I mean, how many times are we going to run across an alternating series? I mean, is Mother Nature so accommodating ...

P: Good question. Let's see what we can do.

Estimating the Sum of "Other" Taylor Series

Consider the Taylor series for e^x, namely $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. If we use it to compute the number "e", by setting x = 1, we'd get the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$ and now we <u>don't</u> have an alternating series. However, we can still estimate the sum of the infinite series as follows:

We consider, again, what the RATIO test is telling us. If $a_n = \frac{1}{n!}$, then $\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \le \frac{1}{10}$ when $n \ge 9$. That means, after $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{9!}$, the terms decrease by at least a factor $\frac{1}{10}$ so the entire balance of the infinite series, namely $\frac{1}{10!} + \frac{1}{11!} + ...$ is less than $\frac{1}{10!}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + ...\right)$ which is a convergent geometric series with $a = \frac{1}{10!}$ and $r = \frac{1}{10}$ so it sums to $\frac{a}{1 - r} = \frac{1}{10!} \frac{1}{1 - \frac{1}{10}} = \frac{1}{9\,(9!)}$. Hence we can see that the sum of the infinite series lies between $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{9!}$ and $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{9!} + \frac{1}{9\,(9!)}$.

S: Hey! That's practically the same as saying the error is less than the first neglected term. Right? I mean, if I stop adding after $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{9!}$, then the infinite series adds up to something between this and $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{9!} + \frac{1}{9(9!)}$ so the error is less than $\frac{1}{9(9!)}$ which is ... uh, *almost* the first neglected term ... which is $\frac{1}{10!}$.

P: The first neglected term is $\frac{1}{10!} = \frac{1}{10(9!)}$ which is only slightly less than $\frac{1}{9(9!)}$, so you're quite right. In fact, if you sum a series $a_1 + a_2 + a_3$ -+ ... and stop with the term a_{99} for example, and if the remaining terms decrease by at least a factor r (according to the RATIO test), then all the remaining terms add up to something less than $a_{100}\,(1 + r + r^2 + ...) = \frac{a^{100}}{1 - r}$ which compares favorably with a_{100}, the first neglected term ... if r is small. So there you have it! A method of estimating the sum of an infinite series (which satisfies the RATIO test), even if every term is positive. Nice, eh?

Example: Estimate the value of $\sqrt{47}$ by using five terms of a Taylor series for $f(x) = \sqrt{x}$, about x = 49.

Solution: We construct a table of derivatives for $f(x) = \sqrt{x}$, evaluated at x = 49 (a place near x = 47 where we actually *know* the value of the function and its derivatives!):

n	0	1	2	3	4
$f^{(n)}(x)$	$x^{1/2}$	$\frac{1}{2}\ x^{-1/2}$	$-\frac{1}{2}\frac{1}{2}\ x^{-3/2}$	$\frac{1}{2}\frac{1}{2}\frac{3}{2}\ x^{-5/2}$	$-\frac{1}{2}\frac{1}{2}\frac{3}{2}\frac{5}{2}\ x^{-7/2}$
$f^{(n)}(49)$	7	$\frac{1}{2\ (7)}$	$-\frac{1}{2^2\ 7^3}$	$\frac{(1)(3)}{2^3\ 7^5}$	$-\frac{(1)(3)(5)}{2^4\ 7^7}$

Now we substitute these derivative values into:

$f(a) + f'(a)\ (x - a) + \frac{f''(a)}{2!}\ (x - a)^2 + \frac{f'''(a)}{3!}\ (x - a)^3 + \ldots$ with a = 49 . That is, we substitute for **( )** in:

$(\ ) + (\ )\ (x - 49) + (\ )\frac{1}{2!}\ (x - 49)^2 + (\ )\frac{1}{3!}\ (x - 49)^3 + (\ )\frac{1}{4!}\ (x - 49)^4 + \ldots$ and this gives:

$\sqrt{x} \approx 7 + \frac{1}{2\ (7)}(x - 49) - \frac{1}{2^2\ 7^3}\frac{1}{2!}\ (x - 49)^2 + \frac{(1)(3)}{2^3\ 7^5}\frac{1}{3!}\ (x - 49)^3 - \frac{(1)(3)(5)}{2^4\ 7^7}\frac{1}{4!}\ (x - 49)^4$ (where we use only five terms) and since we want to estimate $\sqrt{47}$ we substitute x = 47 and get:

$\sqrt{47} \approx 7 - \frac{1}{7} - \frac{1}{7^3\ 2!} - \frac{3}{7^5\ 3!} - \frac{(3)(5)}{7^7\ 4!}$. Every term is negative (!#$?*& so we don't have the simplicity of an alternating series for estimating our error), but fortunately we only wanted the five-term estimate and that's

$\sqrt{47} \approx 7 - 0.14285714 - 0.00145773 - 0.00002975 - 0.00000076 = 6.85565462$ and although we have no error estimate (without going through the Ritual of the Ratio) we have confidence that this estimate is accurate to perhaps six or seven places of decimals (since the terms are decreasing rapidly).

S: So, what does **MAPLE** say?

P: I'll ask **MAPLE** for a bunch of digits, then to **evalf** the **sqrt** of 47:

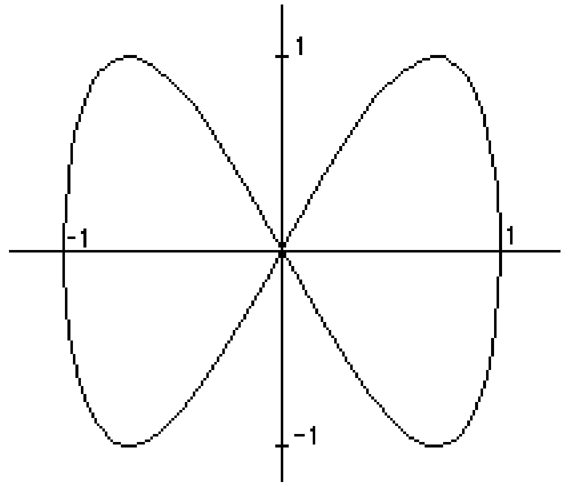
```
• Digits:=75;
                              Digits := 75
• evalf(sqrt(47));
 6.85565460040104412493587144908484896046064346100132627548510818567851711514
```

S: Not bad.

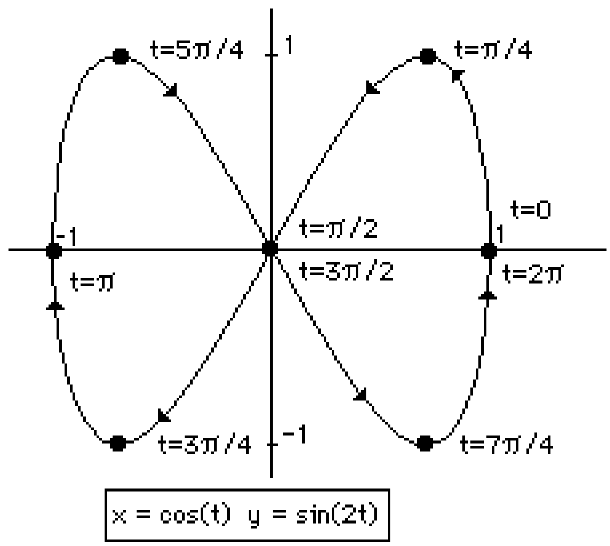
LECTURE 10

CURVES and PARAMETRIC EQUATIONS

Way back when, we defined a function:
f(x) is a function if, for each value of x in some "domain", there is a single, unique value of f(x) defined.
What this means is that the graph of $y = f(x)$ must satisfy the "vertical line test". That is, every vertical line $x = c$ (where c is in the domain) intersects the graph precisely once.
Sad.
How, then, do we describe a curve ===>>
In fact, how do we describe, in terms of "functions", a circle or an ellipse which clearly don't satisfy the vertical line test?
That's what we'll do now:



We describe the location of a point (x,y) via equations such as $\boxed{x = \cos t,\ y = \sin(2t)}$. That means, for each value of "t" (called the "parameter"), there is a single, unique value of x and y (because cos t and sin (2t) are functions, hence provide unique values for each t). The diagram indicates the path of the point (x,y) as t changes from t = 0 through $t = \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}$, etc. until, finally, $t = \frac{8\pi}{4} = 2\pi$ and we've completed the curve. Thereafter, for $t > 2\pi$, the curve repeats itself.



For this particular curve, described by so-called PARAMETRIC EQUATIONS: $x = \cos t$, $y = \sin(2t)$, we can see that $-1 \le x \le 1$, $-1 \le y \le 1$ so the entire curves lies in a square centred at the origin. We also see that x goes from 1 to 0 to - 1 to 0 to 1 and, at the same time, y oscillates between these same values, but twice as rapidly (because $y = \sin(2t)$ oscillates twice as rapidly as $x = \cos t$), so that when x goes through one complete cycle, y goes through *two* cycles. Further, we can easily pick out the maximum and minimum values of x and y, and the values of t where they occur. This gives us enough information to sketch the graph (although the diagram, above, was actually generated by a computer ... just so you could see the precise behaviour).

In general, parametric equations for a curve in the x-y plane are given by: $x = f(t)$, $y = g(t)$ where "f" and "g" are functions of the parameter "t". It's convenient to use the letter "t" as the parameter and to think of it as being the *time*. Then $x = f(t)$, $y = g(t)$ describes the location of a moving point at any time t. It's something like giving the latitude and longitude of an automobile moving along some race track ... or any curve.

It's important to be able to construct parametric equations for some simple curves, and we'll do that now:

Example: Verify that $x = a \cos t$, $y = a \sin t$ are parametric equations for the circle $x^2 + y^2 = a^2$. Then construct parametric equations for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: If $x = a \cos t$, $y = a \sin t$, then $x^2 + y^2 = (a \cos t)^2 + (a \sin t)^2 = a^2 (\cos^2 t + \sin^2 t) = a^2$, as required. For the ellipse we want to find functions x(t) and y(t) so that $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, but this would be the case if $\frac{x}{a} = \cos t$ and $\frac{y}{b} = \sin t$ and that gives parametric equations for an ellipse: $x = a \cos t$, $y = b \sin t$.

S: Are they the only parametric equations for a circle? And how would I know that $x = a \cos t$, $y = a \sin t$ is a circle, if you didn't tell me? And what's the procedure for finding parametric equations for some curve $y = f(x)$, and how would I ...

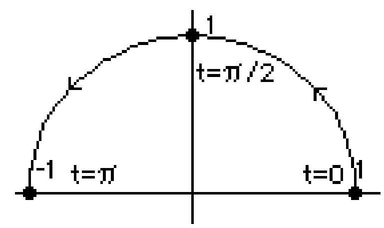
P: Okay, listen. First, if you're given parametric equations like $x = a \cos t$, $y = a \sin t$ and you want to find the equation for this curve as a relation between x and y (without any t's), then you must eliminate the parameter "t". In this case it's pretty easy: $x^2 + y^2 = a^2$: a circle. For $x = a \cos t$, $y = b \sin t$, you'd eliminate "t" by noting that $\cos^2 t + \sin^2 t = 1$ so you'd write $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t = 1$: an ellipse. On the other hand, if you already have $y = f(x)$ and want parametric equations, you can always write $x = t$, $y = f(t)$. See? If you eliminate t between $x = t$ and $y = f(t)$ it'd give $y = f(x)$. Nice, eh?

S: So for a circle, like maybe $x^2 + y^2 = a^2$, I'd just write $x = t$ and $y = ...$ uh, $y =$ *what*?

P: If you tried to solve for y, from $t^2 + y^2 = a^2$, you'd get $y = \pm\sqrt{a^2 - t^2}$ and you wouldn't have a function of t because there are *two* y-values, so that's no good. In fact, what I said was, if you had $y = f(x)$, a function of x, and you wanted parametric equations, then you could let $x = t$ and $y = f(t)$. But you have to start with $y = f(x)$, not something like $x^2 + y^2 = a^2$.

S: Okay, suppose I take $y = \sqrt{a^2 - x^2}$ instead. In case you didn't notice, that's the upper half of the circle, and it is a function. Now I'd let $x = t$ and get $y = \sqrt{a^2 - t^2}$ and, according to you, they'd be parametric equations for this circle, right? But they aren't the same as $x = a \cos t$, $y = a \sin t$! How do like that!

P: First off, if you let "t" take on any value, then $x = a \cos t$, $y = a \sin t$ gives a point which runs around the entire circle, not just the upper half. However, if you restrict t to lie in, say, $0 \le t \le \pi$, then $x = a \cos t$, $y = a \sin t$ do give you parametric equations for the upper half of the circle, and yes, they are different than the parametric equations $x = t$ and $y = \sqrt{a^2 - t^2}$ ... and that's the answer to your first question.



Graph of $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$

S: Huh?

P: You asked if they were the only parametric equations for a circle, and the answer is NO. There are lots of parametric equations for a curve given by, say, $y = f(x)$: $x = t$, $y = f(t)$ is just one of them. You could also let $x = t^3$ and $y = f(t^3)$ since eliminating "t" still gives $y = f(x)$.

S: Hey! Let me do one. I'd let ...

P: Wait, let me give you a particular curve. Try to find parametric equations for the parabola $y = x^2$.

S: Okay, I'd let $x = t$ so $y = t^2$. How's that?

P: Great! Give me another.

S: Okay, I could also let $x = t^2$ and then $y = x^2 = t^4$. I mean, $x = t^2$, $y = t^4$ are also parametric equations for $y = x^2$ and I could also let $x = ...$

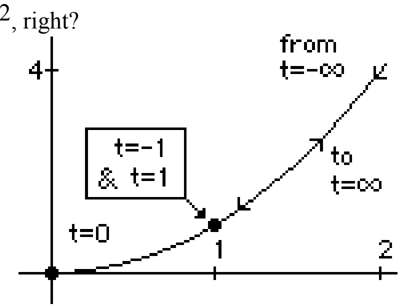
P: One minute. When you let $x = t^2$ you're also inadvertently restricting x to be positive. After all, regardless of the value of the parameter "t", $x = t^2$ will give you a positive value of x. In fact, if you let t run from $-\infty$ to $+\infty$, where will the point (x,y) move?

S: I haven't the foggiest. Uh, wait ... the point will move along the parabola $y = x^2$, right?

P: Yes, but it can never get to the left-half of this parabola because that requires a negative x which you can't get from $x = t^2$.

S: A picture is worth ...

P: Okay, here's the picture ===>>>
See? As x runs from $t = -\infty$ to $t = +\infty$, the point (x,y) runs down the parabola to the origin (which is reached when $t = 0$, because then $x = t^2$ and $y = t^4$ are both 0), then runs back along the same path again, back into the first quadrant. That's why I picked $x = t^3$, $y = f(t^3)$ as parametric equations for $y = f(x)$ because now we'd get the whole

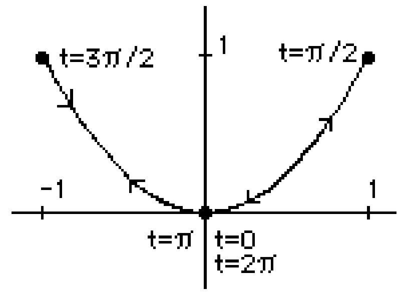


curve, because as t runs from -∞ to +∞, so does x.

Graph of $x = t^2$, $y = t^4$

S: Okay, let me try x = sin t and $y = x^2 = \sin^2 t$. That's a parabola too, right?

P: Yes, but not all of it because x = sin t can only lie between -1 and +1, so you'd get only *that* piece of the parabola.
Before you ask, here's a picture ====>>
See? As t varies from -∞ to +∞, x = sin t just moves back and forth from -1 to 1 to -1 to 1 and so on, while $y = \sin^2 t$ moves back and forth from 0 to 1.



Graph of x = sin t, $y = \sin^2 t$

The curve $y = f(x)$ with domain $a \le x \le b$

has parametric equations $x = t,\ y = f(t)$ with $a \le t \le b$

When I was young and foolish, I wrote a computer program which would plot the graph of y = f(x). The program asked the user to type in f(x) and the domain of x, and then it plotted y = f(x). The program could not plot circles because there was no way my program would accept $y = \pm\sqrt{1-x^2}$ as the function ... smart program! (It's *not* a function.) Then I ate my smart pills and changed the program to plot curves given by *parametric* equations. The user was asked to type in *two* functions, f(t) and g(t), and the domain of t, and the program plotted x = f(t), y = g(t). NOW I could plot circles (as well as other wonderful curves) ... smart program!
If you wanted to plot y = sin x, you typed in the two functions t and sin t and the program plotted x = t, y = sin t. In general, for y = f(x), you gave the program x = t and y = f(t).
But there was a bonus in this modified program. If you wanted to plot r = f(θ), a POLAR curve, you just remembered that x = r cos θ, y = r sin θ, hence x = f(θ) cos θ and y = f(θ) sin θ, so you just typed in the two functions f(t) cos t and f(t) sin t and the program plotted x = f(t) cos t, y = f(t) sin t. Of course, it didn't matter that the name of the parameter was "t" instead of "θ".

The POLAR curve $r = f(\theta)$ has parametric equations $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$

Sketching curves given by parametric equations is sometimes more difficult than curves described by y = f(x) --- and sometimes less difficult. We'll do some examples:

Plotting Parametric Curves

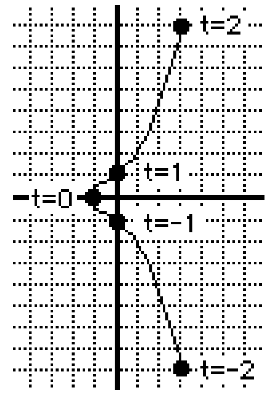
Example: Sketch $x = t^2 - 1$, $y = t^3$.

Solution: We could make a table of values, like so:

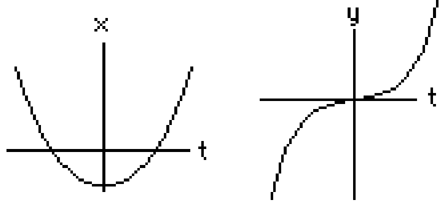
t	-2	-1	0	1	2	3	etc.
x	3	0	-1	0	3	8	etc.
y	8	1	0	1	8	27	etc.

and then plot points, one-by-one (perhaps keeping track of what points go with what t-value by writing the t-value beside the point).

We'd get something like this ===>>>

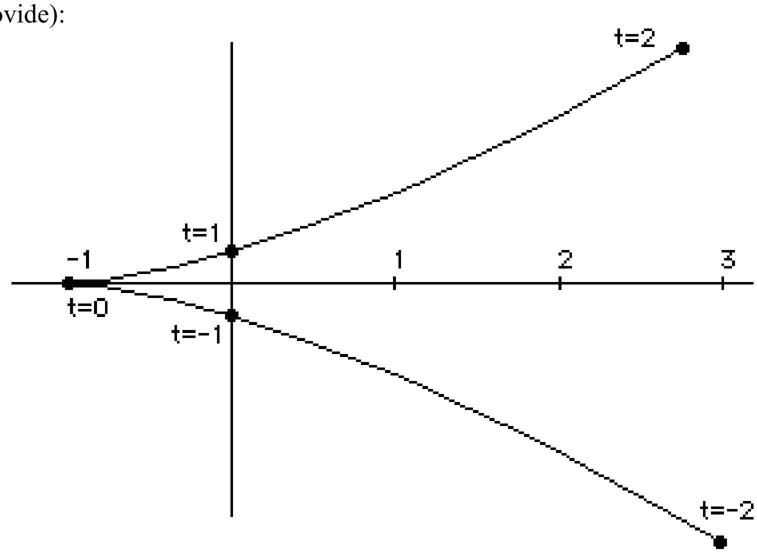


We might also make tiny sketches of $x = t^2 - 1$ and $y = t^3$ like so:



Then we'd note that as t goes from $-\infty$ to $+\infty$, x begins at $+\infty$ and decreases to -1 (at t = 0) then increases again to $+\infty$. In the meantime, y starts at $-\infty$, increases to 0 (at t = 0) and continues to $+\infty$. The point (x,y) then begins in the 4^{th} quadrant at $(+\infty,-\infty)$ then moves up and to the left (x decreasing, y increasing) until it hits (-1,0) at t=0, then x increases again as does y, and the curve heads into the 1^{st} quadrant, heading for $(+\infty,+\infty)$. That's enough to get a sketch of the curve. In fact, it isn't even necessary to plot $x = t^2-1$, $y = t^3$. If you can visualize what they look like, that's enough.

Now let's get my computer program to provide an accurate plot (... well, the accuracy is more a function of what my printer can provide):



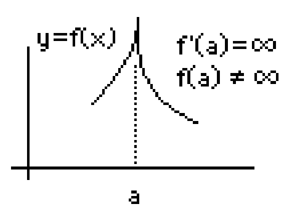
Surprise! The curve has a cusp at t = 0!! Something we might not have expected, and probably wouldn't have caught had we been satisfied with "sketches" based upon a few points, or even arguments like "x goes West and y goes North, then x goes East and y continues North". What we <u>really</u> need is ...

S: Wait! Don't tell me! We need to ... uh, we need ... uh, what do we need?

P: If you had to sketch y = f(x) ... forget about parametric equations for the moment ... and you wanted to catch any "cusps" where the curve had a sharp point (not a "corner", but a "point") then you'd look for places where $\frac{dy}{dx} = \infty$, but y wasn't. Right?

S: Huh?

P: Let's look at a typical cusp ===>>
See? The derivative becomes infinite, but y is finite. On the other hand, if y becomes infinite as well, you wouldn't have a cusp, you'd have a vertical asymptote.



S: What about if the cusp is horizontal. I mean, what if the point is pointing East or West?

P: The y = f(x) wouldn't be a function, would it? I just want to consider what happens when a cusp occurs, for a "function" y = f(x). See? To identify a cusp we have to consider the derivative and that brings us to the next step in parametric equations: derivatives!

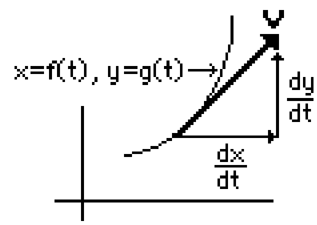
S: But can't we do some more plotting, or sketching, or whatever?

P: Derivatives will help, just as they did for y = f(x).

Slope of the Tangent Line to a Parametric Curve

Consider (again) $x = t^2-1$, $y = t^3$. Note that $\frac{dx}{dt} = 2t$ tells us when x is increasing (when 2t>0) or decreasing (when 2t<0). Also, $\frac{dy}{dt} = 3t^2$ is never negative so y never decreases. I say "**when** x is increasing" because I like to think of the parameter "t" as time. In fact, if x is measured in metres and t in seconds, $\frac{dx}{dt}$ is measured in metres/second, a velocity, and it's the velocity with which the point (x,y) moves left-right (or East-West). On the other hand, $\frac{dy}{dt}$ is the up-down (or North-South) velocity. That's nice. For those of you who are familiar with **vectors**, the velocity vector **V** of the moving point (x,y) has two components, an x-component $\frac{dx}{dt}$, and a y-component $\frac{dy}{dt}$ and we can denote this by writing $\mathbf{V} = \left[\frac{dx}{dt}, \frac{dy}{dt}\right]$. Since the point moves in the direction of the curve itself (else it wouldn't follow the curve!), the velocity vector must point in the direction of the curve itself. That means that the vector **V** is tangent to the curve at every instant of time "t", hence must point along the tangent line. That means that we can find the slope of this tangent line just by finding the slope of **V** and that means ...

S: Whoa! A picture is worth ...
P: Here's the picture ===>>>



Notice that the vector **V** has a slope of $\frac{\text{y-component}}{\text{x-component}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and that's how we can find the slope of the tangent line to a curve given parametrically.

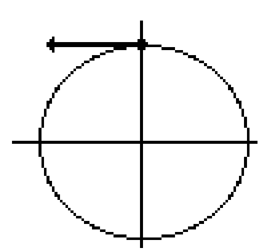
The slope of the tangent line to x = f(t), y = g(t) is $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$

Example: Determine the slope of the tangent to each of the following at the indicated value of the parameter. In each case, sketch the curve and the velocity vector at the indicated point.

(a) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{2}$ (b) $x = t^2-1$, $y = t^3$ at t = 2.

Solution:

(a) $\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t = 0$ at $t = \frac{\pi}{2}$.



Note that, at $t = \frac{\pi}{2}$, $\frac{dx}{dt} = -\sin t = -1$ meaning the point is moving left and $\frac{dy}{dt} = \cos t = 0$ meaning the point is NOT moving up or down. At the point (x,y) = (cos t, sin t) = (0,1), when $t = \frac{\pi}{2}$, the velocity vector is **V** $= [-\sin t, \cos t] = [-1, 0]$ as shown,

(b) $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t^2}{2t} = \dfrac{3}{2}\ t = 3$ at t = 2.

When t = 2, we're at the point $(2^2-1,2^3) = (3,8)$ and the velocity vector has components $\frac{dx}{dt} = 2t = 4$ (to the right) and $\frac{dy}{dt} = 3t^2 = 12$ (upward) ... as shown, i.e. $\mathbf{V} = [4,12]$ when t = 2.

S: Hey, what about that cusp we were talking about?

P: Oh, yes ... let's discuss that. For $x = t^2-1$, $y = t^3$, we have the slope $\frac{dy/dt}{dx/dt} = \frac{3}{2}\ t$ as obtained above. That means that, at t = 0, the slope is zero so the point (x,y) is travelling horizontally.

S: Hold on! Since $\frac{dx}{dt} = 2t = 0$, it isn't travelling horizontally at all. In fact, $\frac{dy}{dt} = 3t^2 = 0$ too, so it isn't even travelling up or down. In fact, the point isn't even moving when t = 0! That's amazing ... isn't it?

P: You surprise me ... that's quite clever of you, and you're quite right, the point comes to a momentary halt at t = 0 before it does an about face and continues again in precisely the opposite direction, heading eventually into the first quadrant.

S: The opposite direction?

P: Sure, look at $\frac{dx}{dt} = 2t$. It's negative for $t<0$ so the point is moving <u>left</u>, then it's 0 when the point stops moving at t = 0, then it's positive when $t>0$ so the point is now moving <u>right</u>. In the meantime, although $\frac{dy}{dt} = 3t^2$ is never negative it <u>is</u> zero momentarily, but then y continues to increase.

S: Hey! If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ then how come the ratio isn't $\frac{0}{0}$? I mean, when you evaluated $\frac{dy/dt}{dx/dt}$ you didn't get $\frac{0}{0}$, you got 0 ... so how come?

P: Another good comment. In fact, I really need to find the limit of $\frac{dy/dt}{dx/dt}$ as $t\to 0$ and that <u>is</u> 0. You see, although $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are both approaching zero, $\frac{dy}{dt}$ is approaching much faster because it's $3t^2$ and $\frac{dx}{dt}$ is only 2t and, when t is small, $3t^2$ is *much* smaller than 2t. See?

S: No.

P: Then forget it ... but remember this: you get much more information by considering $\frac{dx}{dt}$ and $\frac{dy}{dt}$ separately than by simply dividing to get $\frac{dy/dt}{dx/dt}$. You see, if both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ were negative, their ratio would be positive and you'd get a positive slope alright ... but the point is actually moving *down and to the left* along the curve, not *up and to the right*. See?

S: No.

P: Then forget it ... and I take back what I said about your being clever.

Example: Calculate each of the following:

(a) The equation of the tangent line to $x = e^t$, $y = \cos t$ at the point (1,1).

(b) $\frac{dy}{dx}$ at x = 1 if $x = t^3$, $y = \arctan t$.

Solution:

(a) If $x = e^t = 1$ then t = 0 and we check to see that $y = \cos 0 = 1$ so t = 0 <u>does</u> place us at the point (1,1). Now

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{e^t} = 0 \text{ when } t = 0. \text{ Hence the tangent line is: } \frac{y-1}{x-1} = 0 \text{ or } y = 1.$$

(b) If $x = t^3 = 1$, then t = 1 and $y = \arctan 1 = \frac{\pi}{4}$ so we're at the point $(1,\frac{\pi}{4})$. Further, $\frac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{1}{1+t^2}}{3t^2} = \frac{1}{6}$ so the tangent line (using the *point-slope* form) is $\frac{y - \frac{\pi}{4}}{x - 1} = \frac{1}{6}$ or $y = \frac{\pi}{4} - \frac{1}{6} +$ **Error!**.

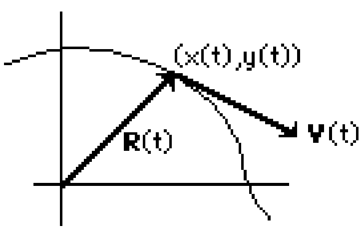
LECTURE 11

MORE ON PARAMETRIC REPRESENTATION OF CURVES

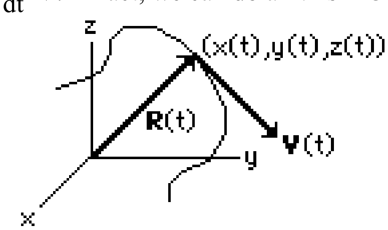
the Tangent and Position Vectors

For a curve described parametrically, x = f(t), y = g(t), the tangent vector is $\mathbf{V} = \left[\frac{dx}{dt}, \frac{dy}{dt}\right] = [f'(t), g'(t)]$. If we think of "t" as being time and "x" and "y" as being distances, then **V** can be interpreted as the velocity vector for a point moving along this curve. In fact, the speed with which the point (x,y) moves can now be obtained: it's the magnitude (or length) of the vector **V**, namely $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Now that we're into vectors, we can also describe the location of the point (x,y) as a "position vector": $\mathbf{R} = [x, y]$ and can think of **R** as a vector which begins at (0,0) and extends to the point (x,y). The length of **R** is, of course, **Error!**. We recognize **Error!**as the distance from (x,y) to the origin (what else?). Let's make all this more prominent:

> For a point (x,y) moving along the curve: x = f(t), y = g(t), we have:
> $\mathbf{R} = [x, y]$ = the position vector, $\mathbf{V} = \left[\frac{dx}{dt}, \frac{dy}{dt}\right]$ = velocity vector, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ = the speed



In the diagram we illustrate a point located at (x(t),y(t)), at time t, and its position vector $\mathbf{R} = [x, y]$ and velocity vector $\mathbf{V} = \left[\frac{dx}{dt}, \frac{dy}{dt}\right]$ which points along the tangent line, in the direction of motion of the point. If, by the phrase "the derivative of a vector **R**" we mean another vector whose components are the derivatives of the components of **R**, then we can write: $\boxed{\mathbf{V} = \frac{d}{dt}\mathbf{R}}$ which is very nice indeed. In fact, we can actually differentiate **V** to get the acceleration, $\mathbf{A} = \frac{d}{dt}\mathbf{V}$. In fact, we can do all this in 3 dimensions too, defining a point (x,y,z) in 3-dimensional Cartesian (or rectangular) coordinates which moves along a curve given parametrically by x = f(t), y = g(t), z = h(t), then introducing the point's position vector $\mathbf{R} = [x, y, z]$ and its velocity vector $\mathbf{V} = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$ and even its acceleration vector $\mathbf{A} = \frac{d}{dt}\mathbf{V} = \frac{d^2}{dt^2}\mathbf{R} = \left[\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right]$ and its distance



from the origin: the length of **R**, namely $\sqrt{x^2+y^2+z^2}$, and its speed, the length of **V**, namely $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$ and so on. In fact, it's now clear that the expression for the "slope of the tangent line" we obtained earlier for curves in the x-y plane, namely $\frac{dy/dt}{dx/dt}$, is really not something that easily generalizes to 3-dimensions ... and that's one reason for keeping your eye on each component $\frac{dx}{dt}$ and $\frac{dy}{dt}$ rather than on their ratio. Indeed, it's not even clear what we would mean by the phrase "slope of the tangent line" for curves in 3-

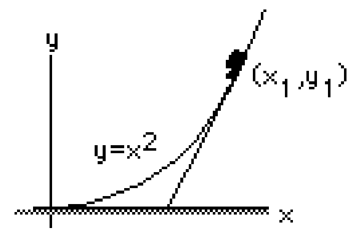
dimensions. (We'll discuss this very problem in later lectures.) What is clear, is that these expressions are just as easily written for 10 dimensions! A point with rectangular coordinates $(x_1, x_2, ..., x_{10})$ has position vector **R** = $[x_1, x_2, ..., x_{10}]$ and if $x_1, x_2, ..., x_{10}$ change with time "t" ... describing a curve in 10-dimensional space, given "parametrically" by, say $x_1 = f_1(t)$, $x_2 = f_2(t)$, ..., $x_{10} = f_{10}(t)$, then it has velocity $\mathbf{V} = \left[\frac{dx_1}{dt}, \frac{dx_2}{dt}, ..., \frac{dx_{10}}{dt}\right]$ which is a vector which points in a direction tangent to the curve, and the speed would be the length of **V**, namely ... well, you get the idea, so let's get back to curves in the x-y plane.

S: Do you remember way back when ... in the previous course ... you were talking about a car moving along a highway and you said you couldn't complete the problem until you had parametric equations to work with ... remember?

P: Yes, vaguely ... let's do that now. It's in your notes. Tell me about it.

S: A car moves south-west along a highway described by the curve $y = x^2$ (where the x-axis points east-west and the y-axis north-south). Its headlights illuminate a fence which lies along the x-axis. Investigate the speed with which the point of light moves along the fence. And you said:

The headlight beam is tangent to the curve $y = x^2$ and will strike the fence (i.e. will intersect the x-axis) at the x-intercept of the tangent line. So we'll pick some point of the curve then we'll find the equation of the tangent line at this point, then we'll find the x-intercept, then we'll find $\frac{d}{dt}$ of this x-intercept. Let the point on the curve be (x_1,y_1) where $y_1 = x_1^2$. Then the tangent line



equation is $\frac{y - y_1}{x - x_1}$ = slope of tangent line = $2x_1$ (since $\frac{dy}{dx} = 2x$ if $y = x^2$). The tangent line intersects the x-axis when y = 0 so we solve for $x = x_1 - \frac{y_1}{2x_1}$. Plug in $y_1 = x_1^2$ and get the x-intercept as $x = \frac{x_1}{2}$ and if we take $\frac{d}{dt}$ of this we get $\frac{dx}{dt} = \frac{1}{2}\frac{dx_1}{dt}$ so the speed with which the light travels along the fence is always half the speed with which the car moves west. Then I said "Suppose the car is moving at 100 km/hour? Then how fast is the light moving?" So? How fast?

P: Okay, we need to parameterize the equation $y = x^2$. Although I had earlier used x_1 and y_1 to represent the point on the curve, I hate to get things cluttered with those subscripts, so now we'll use x and y for the point on the curve. Okay, for parametric equations we could say $x = t$, $y = t^2$ but if x and y are measured in km and t in hours, then $\frac{dx}{dt}$ = 1 km/hour and that doesn't change and there's nothing we can do about that. We clearly need a parametric description, $x = f(t)$, $y = g(t)$, so that the speed is constant at 100 km/hour. And what's the expression for the speed?

S: Uh ... $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.

P: Right! So we need to find functions x(t) and y(t) ... can I call them that? Thanks. I hate to use f(t) and g(t). I'd much prefer ...

S: Don't digress.

P: Okay, we need to find two functions, x(t) and y(t), and these two functions must be such that

(1) $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ = 100 km/hour, and

(2) the point (x(t),y(t)) is on the curve $y = x^2$.

To do this, we use (2): $y = x^2$, so that $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ (remember the Chain Rule?) $= 2x\frac{dx}{dt}$ so we substitute into (1) to get $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(2x\frac{dx}{dt}\right)^2} = 100$ or (factoring out the $\frac{dx}{dt}$ and taking its root) $\sqrt{1+4x^2}\,\frac{dx}{dt} = 100$ so that the function x(t) must satisfy this differential equation. After solving it, we can find y(t) since it's just the square of x(t). Do you remember how to solve such a DE?

S: No problem. It's separable so I just collect everything involving x together with the dx and stick everything else on the other side ... uh, that gives me $\sqrt{1+4x^2}\,dx = 100\,dt$ so then I integrate and get $\int\sqrt{1+4x^2}\,dx = 100t + C$ and ... uh, can I do that one?

P: No, but I'll let you look it up in a table of integrals. Here's one.

$\int \frac{\sqrt{2ax - x^2}}{x} dx = \sqrt{2ax - x^2} + a \arcsin \frac{x - a}{a}$	$\int \sqrt{2ax - x^2} dx = \frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2} \arcsin \frac{x - a}{a}$
$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln \lvert x + \sqrt{x^2 \pm a^2} \rvert$	$\int x^2 \sqrt{x^2 \pm a^2} dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln \lvert x + \sqrt{x^2 \pm a^2}$

S: Okay ... uh, it's not there! What kind of table ... ?

P: Here it is: $\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln \lvert x + \sqrt{x^2 \pm a^2} \rvert$

S: Are you kidding? That's not the same integral! I mean ...

P: Pay attention. Your integral is $\int \sqrt{1+4x^2} dx$ and you can easily make it look like the tabulated integral by letting $u = 2x$ so $du = \frac{du}{dx} dx = 2 dx$ then you'd get $\frac{1}{2}\int \sqrt{1+u^2} du = \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2} \ln \lvert u + \sqrt{u^2+1} \rvert\right)$ where I used the integral in the tables with a = 1 and the + sign and, of course, calling the variable "u" instead of "x", so I'd get $\int \sqrt{1+4x^2} dx = \frac{x}{2}\sqrt{4x^2+1} + \frac{1}{4} \ln \lvert 2x + \sqrt{4x^2+1} \rvert$ replacing u by 2x. Okay, carry on.

S: Huh? Oh, yeah ... I just solved the DE and it's $\frac{x}{2}\sqrt{4x^2+1} + \frac{1}{4} \ln \lvert 2x + \sqrt{4x^2+1} \rvert = 100 t + C$. Wow! Is that the curve? I mean, wow!

P: No, that's not the curve. The curve is $y = x^2$. What you've found is how x must change with time in order that the speed, along this parabolic highway, is constant at 100 km/hour. Since we only asked for how rapidly the headlight beam is moving along the fence, and since we know it moves at $\frac{1}{2}\frac{dx}{dt}$ and we know that $\sqrt{1+4x^2}\frac{dx}{dt} = 100$, then the speed along the fence is $\frac{50}{\sqrt{1+4x^2}}$ and we have the speed of the headlight.

S: We do?

P: Sure! It changes with time, of course, but if you want to know the speed when x = 1 km, it's $\frac{50}{\sqrt{1+4}}$ km/hour, and so on. If you know the position of the car, meaning you know x, then you know the speed of the beam along the fence. See? If you want to know the speed at some particular time, you'd plug t into $\frac{x}{2}\sqrt{4x^2+1} + \frac{1}{4} \ln \lvert 2x + \sqrt{4x^2+1} \rvert = 100 t + C$ and solve for x, but then you'd first have to know the value of C and that means you'd have to give me one piece of information about the location of the car at some time, say at t = 0, so I could find C. See?

S: I wish I hadn't brought the subject up.

P: Do you see anything unusual about the speed of the beam: $\frac{50}{\sqrt{1+4x^2}}$?

S: When x = 0 the car has reached the fence and the beam speed is ... uh, 50 which is half the car speed, just like it should be. Good, eh?

P: Yes, good, but do you see anything else ... unusual ... about this beam speed?

S: Nope.

P: It's positive, meaning ...

S: It's moving to the right, right?

P: Right! And since that's not possible, we must ... uh, you must have made a mistake. Where's your mistake?

S: Huh?

P: You had $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(2x\frac{dx}{dt}\right)^2} = 100$ then you factored the $\frac{dx}{dt}$, took the root and got $\sqrt{1+4x^2}\frac{dx}{dt} = 100$ when you should have taken the root of $\left(\frac{dx}{dt}\right)^2$ as $-\frac{dx}{dt}$ giving $-\sqrt{1+4x^2}\frac{dx}{dt} = 100$ meaning $\frac{dx}{dt} < 0$ so the car was moving West. See? Then you'd have $\frac{dx_1}{dt} < 0$ too. See?

S: I don't remember doing that. Hey! <u>YOU</u> did that!

P: Hmmm ... let's go on.

Length of a Curve

P: Now that we've returned to the car on the highway, we can comment that the speedometer of the car measures ... what?

S: The speed, and that's $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$.

P: And the odometer? What does that measure?

S: The mileage. I mean, how far you've gone ... the distance.

P: And are the speed and distance travelled ... are they related?

S: Uh ... if s is the distance, then $v = \frac{ds}{dt}$, right?

P: Right! The distance travelled is s, shown on the odometer. The speed is v, shown on the speedometer, and $v = \frac{ds}{dt}$ so the speedometer reading is actually the derivative of the odometer reading.

S: Wow! My car does that?

P: Sure, every car has taken this course and ...

S: Don't digress.

P: Okay, pay attention:

If "s" measures the distance travelled along some curve, given parametrically by x = x(t), y = y(t), and v is the speed, namely the length of the vector $\mathbf{V} = \left[\frac{dx}{dt}, \frac{dy}{dt}\right]$, then $v = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$. This actually allows us to compute the distance travelled if x(t) and y(t) are known (meaning the position, at any time t), since we'd know $\frac{ds}{dt}$ and we could then integrate to find s.

Example: A point moves according to x(t) = 10 cos t, y(t) =10 sin t, where "x" and "y" are distances, in km, and "t" is the time, in hours. How far does it move during the time interval from t = 0 to t = 5?

Solution: $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{100 \sin^2 t + 100 \cos^2 t} = \sqrt{100} = 10$ km/hour which gives a simple DE for s: solving $\frac{ds}{dt} = 10$ yields: s(t) = 10t + C. But s = 0 when t = 0 so 0 = 0 + C, hence C = 0 and s = 10 t km, so after 5 hours, the point moves 50 km ... OR, we could have written, simply, $s = \int_0^5 10\, dt = [10t]_0^5 = 50$ km.

Note: Since $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$, we can find the length of the curve, from t = a to t = b, from

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

Example: A point moves according to x(t) = t, $y(t) = \frac{e^t + e^{-t}}{2}$, where "x" and "y" are in metres, and "t" is in seconds. How far does it move during the first second?

Solution: $s = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(1)^2+\left(\frac{e^t - e^{-t}}{2}\right)^2}\, dt = \int_0^1 \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4}}\, dt$

$$= \int_0^1 \sqrt{\left(\frac{e^t + e^{-t}}{2}\right)^2}\, dt = \int_0^1 \frac{e^t + e^{-t}}{2}\, dt = \frac{1}{2}\left[e^t - e^{-t}\right]_0^1 = \frac{1}{2}(e - e^{-1}) - \frac{1}{2}(1 - 1) = \frac{e - e^{-1}}{2} \text{ metres.}$$

S: Hey! That's pretty tricky! I mean, you rearrange and get a perfect square and take the square root and ...
P: Calculus instructors stay awake nights thinking of such examples. Pay attention.

LECTURE 12

SOME APPLICATIONS

Polar Curves, revisited

We saw that the equation of a polar curve, $r = f(\theta)$, can be put into parametric form by writing $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ where the parameter θ is in fact the angle θ. Because of this we can use parametric techniques for finding $\frac{dy}{dx}$, the slope of the tangent line to a polar curve and the length "s" of a polar curve.
Recall the magic formulas for a parametric curve given by: $x = x(t)$, $y = y(t)$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For the slope of the tangent to our POLAR curve $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ (where the parameter is called θ, not t) we use:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f(\theta)(-\sin\theta) + f'(\theta)\cos\theta} \quad \text{which can also be written} \quad \boldsymbol{\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}}$$

(putting $f(\theta) = r$ and $f'(\theta) = \frac{dr}{d\theta}$).

For the length of a POLAR curve we first compute $\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$. Putting $\frac{dx}{d\theta} = f(\theta)(-\sin\theta) + f'(\theta)\cos\theta$ and $\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$ and squaring and adding we note (surprise!) that there is some cancellation and we can use $\cos^2\theta + \sin^2\theta = 1$ and we get, simply $\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(f(\theta))^2 + (f'(\theta))^2}$ so the length of the polar curve, from $\theta = \alpha$ to $\theta = \beta$ is given by:

$$s = \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta \quad \text{or} \quad \boldsymbol{s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example: For the polar curve $r = \sin(2\theta)$, determine the slope at $\theta = \frac{\pi}{4}$. Also, express as a definite integral the length of the curve from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

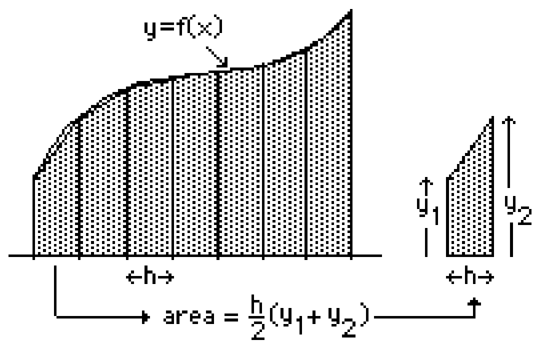
Solution: We have $r = \sin(2\theta)$ so $x = r\cos\theta = \sin(2\theta)\cos\theta$ and $y = r\sin\theta = \sin(2\theta)\sin\theta$, hence

$\frac{dx}{d\theta} = -\sin(2\theta)\sin\theta + 2\cos(2\theta)\cos\theta$ and $\frac{dy}{d\theta} = \sin(2\theta)\cos\theta + 2\cos(2\theta)\sin\theta$ hence the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin(2\theta)\cos\theta + 2\cos(2\theta)\sin\theta}{-\sin(2\theta)\sin\theta + 2\cos(2\theta)\cos\theta} = \frac{(1)(1/\sqrt{2}) + 0}{-(1)(1/\sqrt{2}) + 0} = -1 \text{ when } \theta = \frac{\pi}{4}.$$

Further, $s = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)}\, d\theta$ is the required length of the curve (also called the "arc length").

S: So why don't you evaluate the integral?
P: I can't ... at least not in terms of known functions.
S: So what would you do if you really wanted an answer, I mean, an actual number?
P: I'd plot a graph of $\sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)}$ and calculate the area under the curve from $\theta = 0$ to $\theta = \frac{\pi}{2}$. For that I could use rectangles ... a Riemann SUM, remember those? Or I could use other methods which are more efficient. People have spent a lifetime finding clever ways of evaluating areas or definite integrals, approximately, to any desired degree of accuracy.
S: For example?
P: Well, suppose I wanted the area under y = f(x) from x = a to x = b. I could pick a bunch of points from "a" to "b", evaluate f(x) at each, join all the points on the curve and find the area under this approximation to the curve.
S: Picture?
P: Okay, here's a picture ===>>>



I've divided the interval into 7 subintervals; I'll let h be the width of each. I then calculate the y-values for the 8 points on the curve; I'll call them $y_1, y_2, \ldots y_8$. The area under the zig-zag approximation is the sum of a bunch of trapezoids and I have a formula for that, namely: (*width*) (*average height*) so the trapezoids would have a total area of

$$\frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \frac{h}{2}(y_3 + y_4) + \ldots + \frac{h}{2}(y_7 + y_8)$$

or simply

$\frac{h}{2}(y_1 + 2y_2 + 2y_3 + \ldots + 2y_7 + y_8)$ and it's clear that I could get better and better approximations by choosing more and more points. See? It's called the **TRAPEZOIDAL RULE**.
S: Let's see you do it for **Error!**
P: For n subintervals, if I call the sum of the trapezoidal areas A(n), then **MAPLE** gives the following:

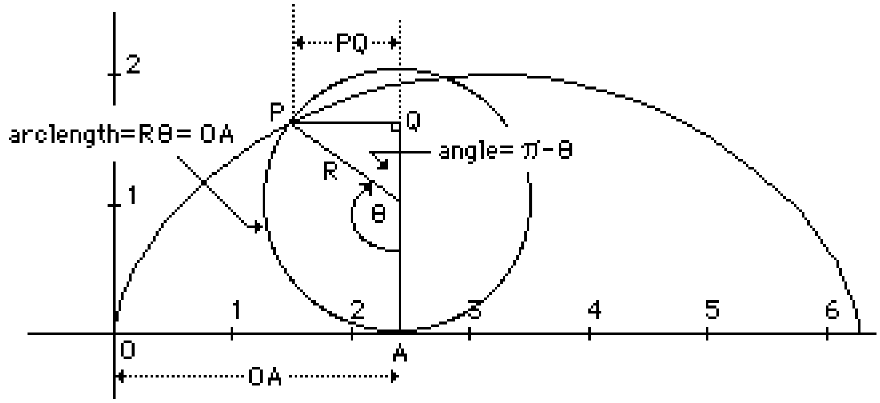
```
A(3)  = 2.432509699
A(5)  = 2.422619019
A(7)  = 2.422145164
A(10) = 2.422111352
A(20) = 2.422112055
A(30) = 2.422112055
```

See? Even a half-dozen trapezoids will give 3 decimal places. Good, eh?

the CYCLOID

Although we introduced parametric equations in order to describe a curve which doesn't satisfy the vertical line test (such as a circle) it is often the case that curves are more easily defined in terms of parametric equations than in the form y = f(x), even in cases where the curve does satisfy the vertical line test (meaning that the curve does represent a function). An example of this is the CYCLOID.
Consider a flashlight attached to the rim of a bicycle wheel. Turn out the lights, roll the wheel and watch the moving light trace out a curve: the cycloid. We'll find the equation of this curve:
We begin with the flashlight located at the bottom of the wheel which we take to be our origin (x=0, y=0), and we roll the wheel (assumed to have a radius R) along the positive x-axis. After having turned through an angle θ

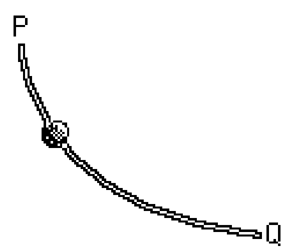
the flashlight is located at some point P(x,y) as shown and we can read off the value of x and y from the diagram, in terms of θ.



$x = OA - PQ = \text{arclength} - R \sin(\pi - \theta) = R\theta - R \sin\theta = R(\theta - \sin\theta)$
$y = R + R\cos(\pi - \theta) = R - R\cos\theta = R(1 - \cos\theta)$

Here we've used $\sin(\pi-\theta) = \sin\pi \cos\theta - \cos\pi \sin\theta = \sin\theta$ and $\cos(\pi-\theta) = \cos\pi\cos\theta + \sin\pi\sin\theta = -\cos\theta$ as well as the fact that the distance 0A is the same as the length of the arc of the circle from A to P which, of course, is Rθ (θ in RADIANS!!!)

S: How did you know that? I mean, the length of the arc ...
P: Well, just trust me.
S: Are you kidding?
P: Okay, it's because I've assumed the wheel rolls without slipping on the ground ... uh, the x-axis. You see, if it slips then it could rotate without the wheel even moving to the right ... it just slips as though the x-axis were a sheet of ice, and the angle θ could be very large because the wheel is rotating but it isn't going anywhere so ...
S: Are you kidding? That's a proof?
P: Well, no, but if it rolls without slipping then ... uh, the arclength is the same as 0A.
S: A snow job, that's what you're giving me. I'm just supposed to swallow that and ...
P: Look, suppose we wrapped a tape around the circumference of the wheel ... sticky tape... so it stuck to the ground and unwound from the wheel as it rotated. Then, after rolling to the position shown in the diagram the ground from 0 to A would have this tape and the arc of the wheel from A to P wouldn't. See? It's the same length of tape. See?
S: No. Besides, what good are cycloids, or are they just playthings for mathematicians to illustrate some parametric stuff.
P: Have I told you the story of the Brachistochrone?
S: Please don't.



P: In 1696, Johann Bernoulli posed the following problem to the mathematicians of the world. Two points P and Q are joined by a wire and a bead is allowed to slide along the wire from P to Q without friction. (Imagine the bead as having a small hole in it so it can slide without falling off.) Now the problem: what is the shape of the wire if the time taken is to be a minimum? Newton solved the problem in a day or so, and so did Leibniz, l'Hopital, Johann himself and his older brother Jakob. The shape is called the "brachistochrone". Guess what it is?
S: Golly gee ... I'd say a CYCLOID.

P: Here's something interesting: The parametric equations for the CYCLOID, $x = R(\theta - \sin\theta)$, $y = R(1 - \cos\theta)$ can, in fact, be solved to obtain a relation directly between x and y since $\cos\theta = 1 - \frac{y}{R}$ means $\theta = \arccos(1 - \frac{y}{R})$ ... if only we had studied the inverse cosine, which we haven't ... so $x = R(\theta - \sin\theta) = R(\arccos(1 - \frac{y}{R}) - \sin(\arccos(1 - \frac{y}{R}))$ but we can simplify $\sin(\arccos(1 - \frac{y}{R}))$ which is just $\sin\theta$ because if $\cos\theta = 1 - \frac{y}{R}$ then $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - (1 - \frac{y}{R})^2}$ so we'd get $x = R\left(\arccos(1 - \frac{y}{R}) - \sqrt{1 - (1 - \frac{y}{R})^2}\right)$ and we could check a few points like putting y = 0 and getting x = R (arccos (1) -

0) = R (0) = 0 which certainly checks and we could put y = 2R (the top of the cycloid) and get x = R (arccos (-1) - 0) = R (π - 0) = Rπ which also checks (it's the length of the circular arc = half the circumference of the circle) and we could also check it dimensionally because if x and R are measured in metres then the right-side should be, too, and it is, because $\frac{y}{R}$ is dimensionless, and so is the angle arccos $(1 - \frac{y}{R})$. See?

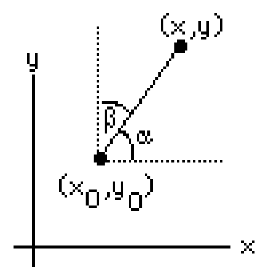
S: zzzzz

the Straight Line

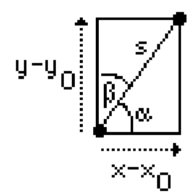
It may seem strange that we choose to find parametric equations for a straight line, but we've already seen that parametric equations for *curves* are just as easy in 3 dimensions or 10 dimensions as in 2 dimensions, so we should be able to deduce parametric equations for 3-D lines by studying, carefully, those for 2-D lines.

We begin with the equation $\frac{y - y_0}{x - x_0} = m$, the *point-slope* form of a straight line (in the 2-dimensional x-y plane).

To get parametric equations we could put $x = t$ and $y = y_0 + m(x - x_0) = y_0 + m(t - x_0)$ and be finished with it. However, it's not so easy to see what this would become in, say, 3 dimensions and one of the nice things about parametric equations is that they don't put any undue strain on x, like "you're the independent variable, remember that!" In fact, writing $x = R(\theta - \sin\theta)$, $y = R(1 - \cos\theta)$ it's clear that x and y share the spotlight ... one is not more or less important than the other ... the equations have a nice symmetry without leaning too heavily on one or the other variable. It's precisely this equality of expression which makes it easy to add more dimensions, so we search for another parametric representation of a straight line which preserves this equality and lack of prejudice, trying to avoid anything which identifies either x or y as being "special". In fact, it's this symmetry and lack of bias that makes $x^2 + y^2 = a^2$ such a nice equation for a circle, as opposed to $y = \sqrt{a^2 - x^2}$.
For our straight line we can keep the point (x_0,y_0) since it doesn't put special emphasis either x_0 or y_0, but we have to forget about the slope "m" since it's the derivative of y with respect to x, or $\frac{\text{the change in y}}{\text{the change in x}}$ and why should *the change in y* be in the attic and *the change in x* in the basement?
Nevertheless, we do want something which indicates the direction of our line, so we choose two angles like α and β in the diagram. They are the angles that the line makes with the positive x- and y-axes.
Now pick any point (x,y) on the line through (x_0,y_0) and let "s" be its distance in the (α,β) direction. The number "s" will be our parameter! That's nice because as the parameter varies and our point moves back and forth along the line, the parameter value actually gives some useful information; it's the distance from (x_0,y_0) to (x,y).
We can identify two right-angled triangles and read off the relationship between x, y, α, β and s:



$x - x_0 = s \cos\alpha$ gives $x = x_0 + s \cos\alpha$
$y - y_0 = s \cos\beta$ gives $y = y_0 + s \cos\beta$



and we have "parametric equations" for our straight line through (x_0,y_0) in a direction determined by angles (α,β):

> The line through (x_0, y_0) which makes angles α and β with the positive x- and y-directions is
> $x = x_0 + s \cos\alpha \qquad y = y_0 + s \cos\beta$

We make several observations:

- $\alpha + \beta = \frac{\pi}{2}$ seems pretty obvious, so $\boxed{\cos\beta = \sin\alpha}$.

- The slope of the line is $\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{\cos\beta}{\cos\alpha} = \frac{\sin\alpha}{\cos\alpha}$ so $\boxed{\frac{dy}{dx} = \tan\alpha}$ ... which is no surprise!
- A line in 3 dimensions, through (x_0, y_0, z_0), making angles α, β and γ with the positive x-, y- and z-axes is:

$$\boxed{x = x_0 + s\cos\alpha,\ y = y_0 + s\cos\beta,\ z = z_0 + s\cos\gamma}$$

Example:

Obtain parametric equations for the line through (1,4) which makes an angle of $\frac{\pi}{3}$ with the positive x-axis.

Solution:

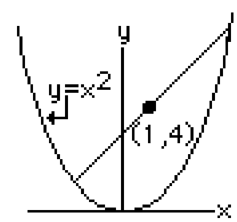
Since $\alpha = \frac{\pi}{3}$, then $\beta = \frac{\pi}{2} - \alpha = \frac{\pi}{6}$ so the equations are: $x = 1 + s\cos\frac{\pi}{3} = 3 + \frac{1}{2}s$, $y = 4 + s\cos\frac{\pi}{6} = 4 + \frac{\sqrt{3}}{2}s$.

Example:

From the point (1,4), how far is it to the curve $y = x^2$ in a direction given by $\alpha = \frac{\pi}{3}$?

Solution: Parametric equations are $x = 1 + \frac{1}{2}s$, $y = 4 + \frac{\sqrt{3}}{2}s$ and since "s" measures distance from (1,4) in the (α,β) direction, we just substitute into $y = x^2$ to find "s": $4 + \frac{\sqrt{3}}{2}s = \left(1 + \frac{1}{2}s\right)^2$ which simplifies to

$s^2 + (4 - 2\sqrt{3})s - 12 = 0$ so $s = \frac{-(4 - 2\sqrt{3}) \pm \sqrt{(4 - 2\sqrt{3})^2 + 48}}{2}$ using the magic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for roots of a quadratic equation. Hence $s = -3.74$ and $s = 3.21$



S: Hey! How can a distance be negative?

P: A picture is worth ... you know. Here's the picture ===>>>

See? The line through (1,4) which makes an angle of $\frac{\pi}{3}$ with the +ve x-axis actually intersects the parabola $y = x^2$ twice! Once in the $\frac{\pi}{3}$ direction and once in the opposite direction ... and you know what that means ...

S: I do?

P: It means that "s" is negative. A positive s-value means the distance in the $\frac{\pi}{3}$ - direction is positive but a negative s-value means ...

S: Yeah, opposite to the $\frac{\pi}{3}$ - direction.

Parametric equations for curves often arise in a quite natural way:

Example: Determine where the line $y = mx$ intersects the curve $\frac{x}{y^2} - \frac{y}{x^2} = 1$.

Solution:

Substituting $y = mx$ we get $\frac{x}{m^2x^2} - \frac{mx}{x^2} = 1$ which can be solved for $x = \frac{m^2}{1 - m^3}$, hence $y = mx = \frac{m^3}{1 - m^3}$ and we have the point of intersection for any value of m (except m = 1). However, if we stand back and stare at what we have, we see that $x = \frac{m^2}{1 - m^3}$, $y = \frac{m^3}{1 - m^3}$ always lies on the given curve $\frac{x}{y^2} - \frac{y}{x^2} = 1$ hence provides (surprise!) parametric equations for that terrible curve. Normally, to obtain parametric equations for this curve we'd put

$x = t$ and try to solve $\frac{t}{y^2} - \frac{y}{t^2} = 1$ for y in terms of t ... good luck!

S: So what happens when m = 1?

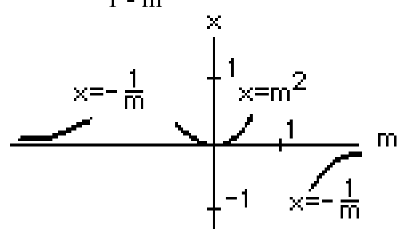
P: Well, let's see what happens when m is close to 1: x and y are both gargantuan. In fact, for m = .9999 say, then both x and y are large but negative so the point of intersection of y = .9999x and $\frac{x}{y^2} - \frac{y}{x^2} = 1$ lies far off in the third quadrant. When m = 1.0001, then x and y are far off in the first quadrant. It seems clear that for m = 1 the line y = x doesn't intersect $\frac{x}{y^2} - \frac{y}{x^2} = 1$ at all.

S: So what does $\frac{x}{y^2} - \frac{y}{x^2} = 1$ look like? I mean, the graph ... what's it like?

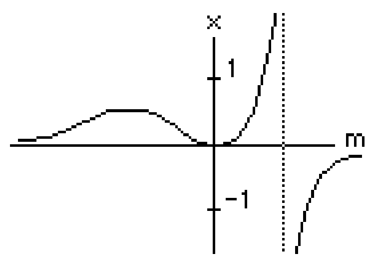
P: Pay attention.

Imagine the problem you'd have if you wanted to plot this graph directly from $\frac{x}{y^2} - \frac{y}{x^2} = 1$. You'd pick a bunch of x-values and for each you'd have to solve for y ... not an easy task. However, we now have parametric equations, so we can easily sketch $x(m) = \frac{m^2}{1 - m^3}$ and $y(m) = \frac{m^3}{1 - m^3}$ as functions of m and determine what happens to the point (x(m),y(m)) as m goes from $-\infty$ to $+\infty$. First we'll do $x = \frac{m^2}{1 - m^3}$, Quick&Dirty.

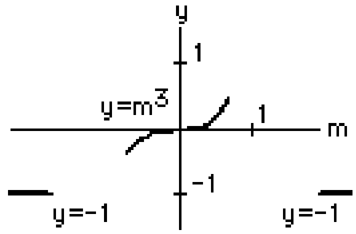
Note that, for m very small, $x \approx \frac{m^2}{1 - 0} = m^2$ (where we ignored the "m^3" compared to the "1") so we sketch that parabola. For m very large, $x \approx \frac{m^2}{0 - m^3} = -\frac{1}{m}$ so we sketch that hyperbola. We also note the vertical asymptote at m = 1 and $\lim_{m \to \pm\infty} x = 0$ (which, of course, is consistent with $x \approx -\frac{1}{m}$ for large m).

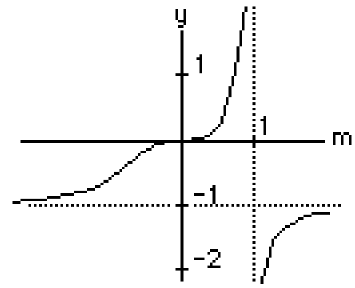


That's enough ... so we complete our sketch of x versus m. Now we do the same sort of thing with $y = \frac{m^3}{1 - m^3}$ noticing that, for small values of m, $y \approx \frac{m^3}{1 - 0} = m^3$ so we sketch that cubic. Then, for m very large, $y \approx \frac{m^3}{0 - m^3} = -1$ so we sketch the line y = -1. We also note the vertical asymptote at m = 1 and that $\lim_{m \to \pm\infty} y = -1$



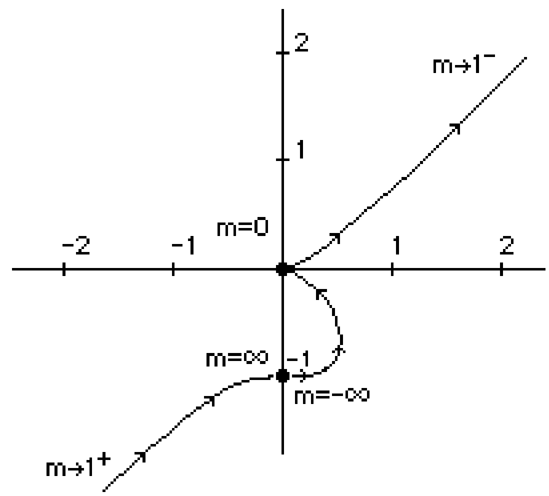
(which, of course, checks with $y \approx -1$ for large m). Both the Quick&Dirty approximations and the final sketch are shown below.





Now we're ready to sketch the curve represented by $x = \frac{m^2}{1 - m^3}$ and $y = \frac{m^3}{1 - m^3}$. We let m begin at $-\infty$ and increase to $+\infty$ and note the following behavior of x and y:

- x begins at 0, increases for a while, then decreases to 0 (when m = 0), then increases to $+\infty$ (as $m \to 1^-$)
- y begins at -1, increases continuously to 0 (when m = 0) and continues to increase to $+\infty$ (as $m \to 1^-$)
  So far, we've got the part of the curve for $-\infty < m < 1$.
  We continue ...
- x begins at $-\infty$ (when $m = 1^+$) and increases continuously to 0 (as $m \to +\infty$)
- y begins at $-\infty$ (when $m = 1^+$) and increases continuously to -1 (as $m \to +\infty$)



Nothing to it!

S: Easy for you to say.

P: You must admit it's easier than trying to plot $\frac{x}{y^2} - \frac{y}{x^2} = 1$.

S: But if you gave me $\frac{x}{y^2} - \frac{y}{x^2} = 1$, how would I know to invent parametric equations by taking lines y = mx and finding the point of intersection ... and all that jazz?

P: You wouldn't ... but then I'm only trying to demonstrate that: (1) parametric equations arise under the most unlikely circumstances and (2) you should be happy when they do and (3) it's often easier to plot curves from the parametric equations and (4) ...

S: Okay, I'm impressed, but I have a question. Even if you gave me $x = \frac{m^2}{1 - m^3}$ and $y = \frac{m^3}{1 - m^3}$ how would I know that "m" was really the slope of some line?

P: Why do you have to know that? When you see x = a cos t, y = a sin t (where "t" is the parameter), you can eliminate t and get $x^2 + y^2 = a^2$ so it's a circle (although it's nice if you recognized the circle directly from the parametric equations) and you don't really need to know *What's the significance of "t"*. In fact, you could think of t as being the time (as I often do ... that's why I like to call it "t"), or you could even notice that it can be interpreted as the angle θ in POLAR coordinates so you might prefer to write x = a cos θ, y = a sin θ and see that it's nobody else but the POLAR curve r = a, a circle of course. In fact, I often wonder what's the significance of the parameter when I run across somebody's parametric equation. On the other hand, sometimes it's quite useful to know "Who's t?".

S: For example?

P: Suppose I gave you the line $x = 1 + \frac{1}{2}t$, $y = 4 + \frac{\sqrt{3}}{2}t$. You might note right away that it's a straight line (I hope you would), but what's the significance of the parameter "t"?

S: I haven't the foggiest ... uh, wait, it's some distance, right?

P: Right. In fact, you can see from these parametric equations that $(x-1)^2 + (y-4)^2 = \left(\frac{t}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}t\right)^2 = t^2$ so "t" gives the distance from (3,4) to (x,y).

S: I have another question. What if I gave you $x = 1 + 2t$ and $y = 4 + 3t$. Then is "t" still the distance?

P: Good question ... in fact, *very* good. Let's check it out: $(x-1)^2 + (y-4)^2 = (2t)^2 + (3t)^2 = 13\,t^2$ so the distance is actually $\sqrt{(x-1)^2 + (y-4)^2} = \sqrt{13\,t^2} = \sqrt{13}\,t$, so "t" is only *proportional* to the distance. In other words, if you really wanted to write the parametric equations in terms of the distance from (3,4), you'd let the distance be "s" and since $s = \sqrt{13}\,t$ you could change the equations to $x = 1 + 2\frac{s}{\sqrt{13}}$ and $y = 4 + 3\frac{s}{\sqrt{13}}$, replacing "t" by $\frac{s}{\sqrt{13}}$ and you'd be happy.

S: Sure, sure ... but I have another question. If you give me $x = 1 + 2\frac{s}{\sqrt{13}}$, $y = 4 + 3\frac{s}{\sqrt{13}}$ (or even $x = 1 + 2t$, $y = 4 + 3t$) then I can see that when $s = 0$ (or $t = 0$) the point is at (1,4) so the line passes through that point. But in what direction?

P: Aah, you should be able to answer that. Try it!

S: Who, me? Uh ... I'd try to find the angle it makes with the positive x-axis, right? But I haven't the foggiest idea ...

P: Pay attention. You could find the slope of the line and that's $\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}}$ or even $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ since it doesn't matter what parametric equations we use. In any case, the slope is $\frac{3/\sqrt{13}}{2/\sqrt{13}} = \frac{3}{2}$ and that's $\tan\alpha$ so $\alpha = \arctan\frac{3}{2}$. Or you could remember the "standard parametric equations" $x = x_0 + s\cos\alpha$, $y = y_0 + s\cos\beta$ where "s" is the distance, then you'd see from $x = 1 + 2\frac{s}{\sqrt{13}}$ that $\cos\alpha = \frac{2}{\sqrt{13}}$ (and of course $\cos\beta = \frac{3}{\sqrt{13}}$ could be obtained from $y = 4 + 3\frac{s}{\sqrt{13}}$).

S: Well, I'd still like to know the significance of a parameter if you give me parametric equations.

P: Remember that parametric equations in a calculus course are usually provided, free of charge. In real life you'd have to generate them yourself and you'd <u>know</u> what the the significance of the parameter was.

The curve $x^{2/3} + y^{2/3} = 1$ is called an Astroid (or 4-cusped hypocycloid). Let's find parametric equations for this curve. First note that it's much like the circle $x^2 + y^2 = 1$ except that the power is 2/3, not 2. But we can take a clue from this since $x = \cos t$, $y = \sin t$ are parametric equations for the <u>circle</u> because $x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$. Now we just modify $x = \cos t$, $y = \sin t$ so the sum $x^{2/3} + y^{2/3}$ is "1" and that means we should take $x = (\cos t)^3$, $y = (\sin t)^3$ because then $x^{2/3} + y^{2/3} = (\cos t)^2 + (\sin t)^2 = 1$. Hence $\boxed{x = \cos^3 t,\ y = \sin^3 t}$ gives parametric equations for an Astroid.

Example: For the Astroid $x = \cos^3 t$, $y = \sin^3 t$, determine:

(1) The equation of the tangent line when $t = A$, putting the equation into a form "symmetrical in x and y"

(2) The x- and y-intercepts of this tangent line.

Solution: We have the slope: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\sin^2 t\cos t}{3\cos^2 t(-\sin t)} = -\tan t$ so that, at $t = A$, the tangent line is

$\frac{y - y(A)}{x - x(A)} = -\tan A$ or $y = y(A) - \tan(A)(x - x(A)) = \sin^3 A - \tan(A)(x - \cos^3 A)$ which is a real mess, so we try for something "symmetrical in x and y". We put $\tan A = \frac{\sin A}{\cos A}$, multiply by $\cos A$, rearrange and get:

$y\cos A + x\sin A = \cos A\sin^3 A + \sin A\cos^3 A = \sin A\cos A(\sin^2 A + \cos^2 A) = \sin A\cos A$ which gives a nice symmetrical equation: $y\cos A + x\sin A = \sin A\cos A$, but we can now divide by $\sin A\cos A$ and get

$\frac{x}{\cos A} + \frac{y}{\sin A} = 1$ which is magnificent! In fact, a straight line in the form $\frac{x}{a} + \frac{y}{b} = 1$ displays, for all to see, the x- and y-intercepts: they're $X = a$ and $Y = b$. For <u>our</u> tangent line, the intercepts are $X = \cos A$ and $Y = \sin A$.

P: See anything interesting about these intercepts?

S: Nope.

P: $X^2 + Y^2 = 1$. Does that say anything to you?

S: Nope.

P: It says that the distance between the intercepts is always equal to 1.

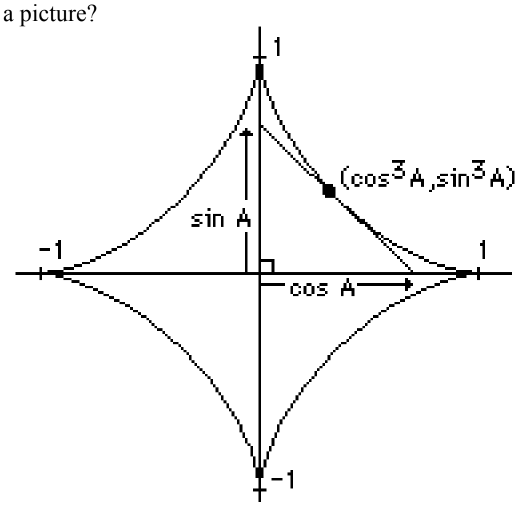
S: So?

P: That's wonderful! Don't you see? It doesn't even matter where you take your tangent line ... what value you give to t = A ... the piece of the tangent line between intercepts is always the same length. Don't you see? The distance is completely independent of A. Pick a different point on the curve and you get the same distance!

S: That's something only a mathie can get excited about. How about a picture?

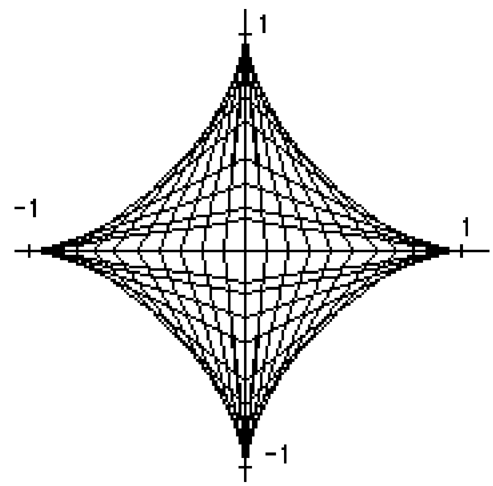
P: Here's a picture ===>>>
A typical point is shown, for some value of A.
The tangent line is drawn. It intersects the axes at cos A and sin A and these change with A, of course, but the distance between them doesn't!

In fact you can try this the next time you're at the beach. Take a stick and draw an x-axis and a y-axis in the sand. Now move the stick so that one end is always on your y-axis and the other is always on your x-axis. The stick will push away the sand and what do you think you'll have ... carved right there in the sand?



S: Picture, please?

P: Okay, the picture ...
But look how nice it was to have parametric equations for the Astroid. You can try it yourself, to prove that the distance between intercepts is a constant independent of where the tangent line is drawn, but try it with $x^{2/3} + y^{2/3} = 1$. Go ahead!



S: You gotta be kiddin'.

P: Okay, then answer this question: what about the distance between intercepts for the curve
$x^{2/3} + y^{2/3} = a^{2/3}$?

S: It's an Astroid too, right? Uh ... I give up. Oh, wait, when the right side was 1 the distance was 1 so that means when the right-side is $a^{2/3}$ the distance must be $a^{2/3}$. Good, eh?

P: Terrible! You haven't learned a thing about dimensions! If x and y are measured in metres and $x^{2/3} + y^{2/3} = a^{2/3}$, then "a" is also in metres, so the distance can't possibly be $a^{2/3}$ which isn't in metres ...

S: I got it! The distance is "a". Am I right?

P: Yes. In fact $x^{2/3} + y^{2/3} = a^{2/3}$ can also be written $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$ so it's just like the last problem except that the variables are $\frac{x}{a}$ and $\frac{y}{a}$ so everything is just scaled by a factor "a" and that means ...

S: Yeah, I get it.