LECTURE 14

AREAS ENCLOSED BY CURVES ... and RIEMANN SUMS

the AREA UNDER A CURVE:

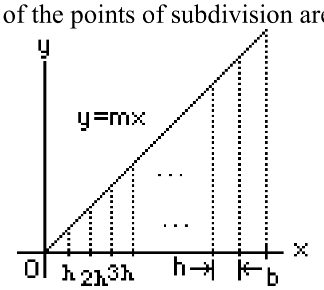
Now we begin the study of the so-called INTEGRAL CALCULUS. Whereas we've been studying DIFFERENTIAL CALCULUS which deals with *differentiation* ... arising from considerations involving the rate of change of functions and the slopes of tangent lines ... NOW we consider *integration* arising from considerations involving the area enclosed by curves.

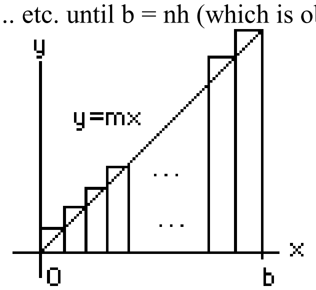
In the *differential calculus* we stood close to a function and investigated its local behaviour ... how it changed when the independent variable changed by some small amount. In the *integral calculus* we stand back and look at the global behaviour ... involving large changes in the independent variables ... and, as we shall see, this is most easily accomplished by summing the effects of small changes!

To begin with, we consider the problem of finding the area beneath a curve $y = f(x)$, from $x = a$ to $x = b$, meaning the area enclosed by the curve $y = f(x)$ and the three lines $x = a$, $x = b$ and $y = 0$ (the x-axis). Later we'll generalize to area enclosed on all sides by curves.

In fact, to simplify even further, we'll consider the area under $y = mx$ from $x = 0$ to $x = b$. Since it's a triangle and we have a formula for that, namely $\frac{1}{2}$ (base) (height) , then we can check the method we're about to describe.

The first step is to divide the interval from $x = 0$ to $x = b$ into n smaller "subintervals" (where n is some large number, like 1,000,000). For the interval $0 \le x \le b$, the n subintervals will have length $h = \frac{b}{n}$ and the x-coordinates of the points of subdivision are 0, h, 2h, 3h, ... etc. until $b = nh$ (which is obvious, since $h = b/n$).





The next step is to erect a rectangle on each subinterval, choosing as height some convenient value of y in each subinterval. (The width of every rectangle is the same: h.) For now, we'll choose as height the value of y at the right-end of each subinterval, meaning mh, m(2h), m(3h) ... etc. (since $y = mx$ gives the height for any x).

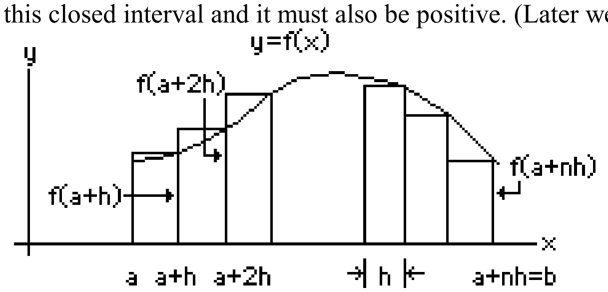
The next step is to add all the areas of the rectangles (which is easy because we have a formula for that). For our choice of heights, we get a SUM of terms each being (width)(height) for each rectangle:

$$\boxed{\text{SUM of areas of rectangles} = h(mh) + h(2mh) + h(3mh) + \dots\ h(nmh)}\ .$$

Finally, we let $n \to \infty$. If the SUM of areas of all the rectangles has a limiting value as $n \to \infty$, then <u>that's</u> the area under the curve. For our problem, we need to find $\lim\limits_{n\to\infty} \big(h(mh) + h(2mh) + h(3mh) + \dots\ h(nmh)\big)$ which seems a formidable task ... but isn't. Remember, m is a constant (the slope of the given straight line) and we can put $\boxed{h = \frac{b}{n}}$. The SUM is then $m\left(\frac{b}{n}\right)^2 (1 + 2 + 3 + \dots + n)$ and there's a magic formula for the sum of the integers from 1 to n, namely $\frac{n(n+1)}{2}$. Hence we can write our SUM of rectangular areas as $\boxed{\frac{m}{2} b^2 \frac{n+1}{n}}$ and find its limit as $n \to \infty$ which is easy since $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$. Finally, then, $\lim\limits_{n\to\infty}$ SUM $= \frac{mb^2}{2} = \frac{1}{2}(b)(mb) = \frac{1}{2}$(base)(height) and (voila!), the technique we've described works for our triangle! In fact, had we taken as height not the value of y at the <u>right</u>-end of each interval but, say, the value at the <u>left</u>-end, we'd get a SUM = $m\left(\frac{b}{n}\right)^2 (0 + 1 + 2 + 3 + ... + (n-1))$

$= \frac{m}{2}\, b^2 \frac{n-1}{n}$ which still has the correct limit: $\frac{mb^2}{2}$. Indeed (remarkably) <u>any</u> y-value in each subinterval could have been chosen as height. The limiting value of the SUM of rectangles would still be $\frac{mb^2}{2}$ ... and that gives us great confidence in this technique. In fact, had we taken some <u>other</u> y-value (rather than the right-y or the left-y which happen to be the maximum-y and the minimum-y in each subinterval) we'd get a SUM of rectangular areas which would be less than the first SUM we obtained, and greater than the second ... and since these two SUMs have the same limit, any <u>other</u> SUM would too (remember the SQUEEZE theorem?).

Now we generalize to an arbitrary $y = f(x)$ on $a \le x \le b$ ... except we must insist that f(x) be continuous on this closed interval and it must also be positive. (Later we'll see what happens when these conditions aren't met.)



First we subdivide the interval $a \le x \le b$ into n subintervals, each of width $\boxed{h = \frac{b-a}{n}}$.

Then we pick some convenient y-value (i.e an f(x)-value) in each subinterval. (In the diagram, we've chosen the value of f(x) at the right-end of each subinterval as we did before.)

Then we construct n rectangles each with width h and height equal to the chosen y-value.

Then we SUM the areas of all rectangles. For our chosen heights, we'd get:

$\boxed{\text{SUM of areas of rectangles} = h\, f(a+h) + h\, f(a+2h) + h\, f(a+3h) + ... h\, f(a+nh)}$.

Finally, we compute $\lim_{n\to\infty}$ (SUM) which we may write (using the sigma-notation for sums):

$\boxed{\text{AREA} = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+kh)\, h}$ and we'd have to compute this limit, realizing that "a" and "b" are constants and $h = \frac{b-a}{n}$. If the limit exists (regardless of which y-values we chose in each subinterval for our heights) then it's the required AREA under y = f(x) from x = a to x = b.

Since h is an increment in x, it's sometimes convenient to use the notation $h = \Delta x$ (as we mentioned when we considered derivatives, writing $f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$). Then our prescription for finding the AREA becomes: $\boxed{\text{AREA} = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\, \Delta x}$.

This limit is called the DEFINITE INTEGRAL of f(x) from "a" to "b" and is written:

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(a+k\Delta x)\, \Delta x = \int_a^b f(x)\, dx}$$

Note: If f(x) is measures in units of *watts* and x is measured in *seconds*, then the SUM $\sum_{k=1}^{n} f(a+k\Delta x)\, \Delta x$ and hence the definite integral $\int_a^b f(x)\, dx$ are measured in *watt-seconds*. If f(x) is in *metres* and x in *degrees*, then the definite integral is measured in *metre-degrees*. This is sometimes useful, especially if f(x) is a rate of change, like *metres per second* and x is measured in *seconds*, then f(x) dx, and hence $\int_a^b f(x)\,dx$, is measured in *metres*. This almost suggests the type of applications one might expect of the definite integral. In fact, let's talk more about these *units* so we get a feel for what the definite integral is doing for us:

- If $\int_a^b f(x)\,dx$ is measured in *kilometres* and x in *hours* then f(x) is in *kilometres per hour* ... a rate of change, so the total number of *kilometres* is obtained by summing the incremental changes in *kilometres* as the *hours* go by, from x = a to x = b.
- If $\int_a^b f(x)\,dx$ is measured in *dollars* and x in *kilograms*, then f(x) is in *dollars per kilogram*... a rate of change, so the total number of *dollars* is obtained by summing the incremental changes in *dollars* as the *kilograms* change, from x = a to x = b.
- If $\int_a^b f(x)\,dx$ is measured in *hectares* and x in *months*, then f(x) is in *hectares per month* ... a rate of change, so the total number of *hectares* is obtained by summing the incremental changes in *hectares* as the *month* go by, from x = a to x = b.

Although we interpret $\int_a^b f(x)\,dx$ as an AREA, it's only so we have some convenient geometrical picture (worth a thousand words?) just as we interpret $\frac{dy}{dx}$ as the slope of some tangent line when, in fact, derivatives are almost never used to determine "slopes". Besides, if f(x) is measured in *metres/second* (a velocity) and x is measured in *seconds*, then f(x) dx (hence the "sum" $\int_a^b f(x)\,dx$) is measured in *metres*... a strange unit perhaps, for AREA! In fact, this so-called AREA actually gives the total distance travelled by an object whose velocity at time "x" is "f(x)".

Example: Compute the total area of n rectangles under y = sin x from x = 0 to x = π.

Solution: We'll subdivide the interval [0,π] into n subintervals of length $\boxed{h = \frac{\pi}{n}}$ and evaluate sin x at x = h, 2h, 3h, ..., nh. This will give the heights of the rectangles. The widths are all the same, namely h. We get:

SUM = h sin(h) + h sin(2h) + h sin(3h) + ... + h sin(nh) or, substituting $h = \frac{\pi}{n}$:

$\boxed{\text{SUM} = \frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \ldots + \sin\frac{n\pi}{n}\right)}$. Now we'll evaluate, to 20 decimal places, for various values of n ... but we'll ask **MAPLE** to do it! We'll: (1) ask for 20 digits, then (2) define h = π/n (**MAPLE** calls π **Pi**), then (3) define our SUM, then (4) **subs**titute various values of n, **eval**uating each time as a **f**loating point (decimal) number (using the **MAPLE** command "evalf"). If we're lucky we'll see the SUM approach some limiting value which is the AREA under the graph of y = sin x from x = 0 to x = π:

```
• Digits:=20;
                              Digits := 20
•h:=Pi/n;
                                h := Pi/n

• SUM:=sum(h*sin(i*h),i=1..n):

• evalf(subs(n=10,SUM));
                          1.9835235375094545037
```

```
• evalf(subs(n=100,SUM));
                                    1.9998355038874435049
• evalf(subs(n=1000,SUM));
                                    1.9999983550656618233
• evalf(subs(n=5000,SUM));
                                    1.9999999342026151656
```

The limit seems to be the number 2, so that's AREA under the graph of y = sin x from x = 0 to x = π.

PS:

P: So let's see you evaluate the definite integral: $\int_0^b mx\,dx$.

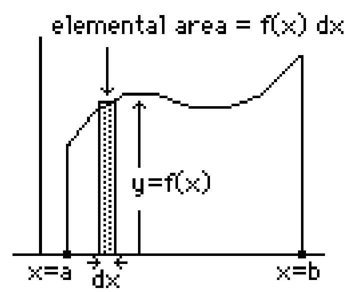
S: Huh?

P: But we've just done that one: $\int_0^b mx\,dx = \frac{mb^2}{2}$.

S: Do you expect me to do all this? I mean, am I expected to find the area by adding up all those rectangles?

P: There's an easier way which we'll get to soon. But remember when we defined the derivative? It was $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ and we actually computed a few derivatives using this definition ... but usually there were easier ways, like the product rule and quotient rule and chain rule. Here, too, we'll seldom have to resort to the definition above. Anyway, how do you like the notation? We start with a SUM, which we write $\sum_{k=1}^{n} f(a+k\Delta x)\,\Delta x$, then we take the limit and get $\int_a^b f(x)\,dx$. See the similarity? It's like starting with the slope of a line joining two points on a curve, $\frac{\Delta y}{\Delta x}$, then taking the limit as $\Delta x \to 0$ and getting $\frac{dy}{dx}$. Nice notation, eh?

In fact I'm told that the symbol $\int$ is an old German symbol for **s** as in **s**um... so you can think of $\int_a^b f(x)\,dx$ as $\int$umming elemental rectangular areas like "f(x) dx". Don't you see? That's the area of a rectangle with height f(x) and width dx. Nice, eh? This is quite nice because you can easily construct the definite integral which represents an area without going through the hassle of finding a Riemann SUM and then ...



S: A who?

P: Oh, I forgot to mention, the SUMs we've been talking about are called Riemann* Sums. That is, $\sum_{k=1}^{n} f(a+k\Delta x)\,\Delta x$ is called a **RIEMANN SUM** and its limit is the **DEFINITE INTEGRAL**. When we found the area under y = mx we first got a Riemann sum: $m\left(\frac{b}{n}\right)^2(1 + 2 + 3 + ... + n)$. The big problem is not to find the Riemann sum, that's just adding up areas of rectangles ... the big problem is finding its limit as $n \to \infty$.

SOME TERMINOLOGY and PROPERTIES of the DEFINITE INTEGRAL:

* Bernhard Riemann (1826-1866) was educated at Gottingen University, Germany, the home of the most gifted mathematician of all time: Carl Friedrich Gauss (1777-1855). Riemann (among many other things!) developed the concept of SUMs (Riemann SUMS ... what else?) as the basis for the definite integral.

In $\int_a^b f(x)\,dx$ we call "a" the LOWER LIMIT and "b" the UPPER LIMIT and f(x) the INTEGRAND.

Because of the way in which the definite integral was defined we have the following properties (which really require a proof ... but we'll just accept them without proof).

(A)	$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$		
(B)	$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$		
(C)	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	(D)	$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
(E)	$\int_a^a f(x)\,dx = 0$	(F)	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

The geometrical meaning of each of the above properties is clear, I think.

(A) The area under $y = f(x) + g(x)$ is the area under $y = f(x)$ PLUS the area under $y = g(x)$.

(B) The area under $y = f(x) - g(x)$ is the area under $y = f(x)$ MINUS the area under $y = g(x)$.

(C) The area under $y = f(x)$ from "a "to "b" PLUS the area from "b" to "c" is the area from "a" to "c".

(D) If the function is multiplied by a constant k, its area is too.

(E) The area from $x = a$ to $x = a$ is zero. (What else?).

(F) Aaah, this one is not so obvious. If $a < b$ (as usual) then the Riemann sum for $\int_b^a f(x)\,dx$ subdivides the interval into subintervals of width $h = \frac{a - b}{n}$ (it's always the *upper* limit minus the *lower* limit) which is negative, so the Riemann sum has the same terms as the Riemann sum for $\int_a^b f(x)\,dx$ ... except for the sign of each term (because $h < 0$). Hence the definite integrals are the negative, one of the other. If that's confusing, then just look at (C) and put $b = a$. You'd get $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$ because of (E) ... and the result is just (F). And if *that's* confusing, then just accept it.

Example: Find a Riemann sum whose limit (as $n \to \infty$) is $\int_1^3 e^x\,dx$.

(Practice the mathspeak: the *lower limit* is 1, the *upper limit* is 3 and the *integrand* is e^x and we're looking at the integral of e^x from 1 to 3 ... NOT the integral "from 3 to 1" ... we've got to get the jargon right!)

Solution: The interval is $1 \le x \le 3$ and we subdivide it into n parts of width $h = \frac{3-1}{n} = \frac{2}{n}$. The points of subdivision are then 1, 1+h, 1+2h, 1+3h, ..., 1+nh (the last of which, of course, is "3"). On each subinterval we construct a rectangle whose height is the value of $y = e^x$ at, say, the left-end. The heights are $e^1, e^{1+h}, e^{1+2h}, \ldots, e^{1+(n-1)h}$. The areas are each (height)(width), namely: $e^1 h, e^{1+h} h, e^{1+2h} h, \ldots, e^{1+(n-1)h} h$ and the Riemann sum is given by: $\boxed{\text{SUM} = e^1 h + e^{1+h} h + e^{1+2h} h + \ldots + e^{1+(n-1)h} h}$. The limit of this SUM, as $n \to \infty$, is $\int_1^3 e^x\,dx$: the area under $y = e^x$ from $x = 1$ to $x = 3$.

PS:

S: Can <u>you</u> find this limit?

P: No problem. Normally I can't, but this one is easy. I factor out $h\,e$ and rewrite the Riemann SUM as $h\,e\,(1 + e^h + e^{2h} + \ldots + e^{(n-1)h})$ and recognize the sum in the brackets as a geometric series for which I have a formula. It sums to ...

S: How do you recognize the sum as a geometric series? I don't.

P: A geometric series has the form $a + ar + ar^2 + ar^3 + \ldots$ where each term, divided by the preceding term, gives the <u>same</u> number "r". In my series, each term divided by the preceding term gives the same number, namely e^h ... so it's a geometric series. The sum of n terms of such a series is $a\,\frac{r^n - 1}{r - 1}$ so <u>my</u> Riemann sum adds up to $h\,e\,\frac{(e^h)^n - 1}{e^h - 1} = h\,e\,\frac{e^{nh} - 1}{e^h - 1}$. I now just have to find the limit of this as $n \to \infty$.

S: Good luck.

P: Remember that $h = \frac{2}{n}$ so $nh = 2$ so $e^{nh} = e^2$ and my SUM can then be written $e\,(e^2 - 1)\frac{h}{e^h - 1}$. Instead of replacing h by $\frac{2}{n}$ and letting $n \to \infty$, I'll leave the h in there and take the limit as $h \to 0$. Our limit is just: $e\,(e^2 - 1) \lim_{h \to 0} \frac{h}{e^h - 1}$ and <u>this</u> limit has the wonderful $\frac{0}{0}$ form so we can use l'Hopital's rule and get: $e\,(e^2 - 1) \lim_{h \to 0} \frac{h}{e^h - 1} = e\,(e^2 - 1) \lim_{h \to 0} \frac{1}{e^h}$ $= e\,(e^2 - 1)\frac{1}{1} = e^3 - e$ (where, to use l'Hopital, we differentiated numerator and denominator with respect to h).

S: Are you saying that the area under $y = e^x$ is $e^3 - e$?

P: From $x = 1$ to $x = 3$ ... yes.

S: Amazing.

P: But there's an easier way. Pay attention:

THE FUNDAMENTAL THEOREM:

In order to avoid finding limits of Riemann sums, we do the following (and whoever thought of this was a genius!). To evaluate $\int_a^b f(x)\,dx$ (and avoid the Riemann summing etc.) we make the upper limit <u>variable</u>, calling it "t". We now have a function of t, namely $A(t) = \int_a^t f(x)\,dx$. The first thing you'd like to do with such a weird and brand-new kind of function is differentiate it! Unfortunately, none of the "differentiation rules" help ... so we must resort to the definition of the derivative. (Imagine the first mathematician who did this. He/she read a book on the Newton/Leibniz differential calculus and decided to use it on this area function, A(t). Imagine the surprise when ... but we get ahead of ourselves.)

Consider $\frac{A(t+h) - A(t)}{h}$ (whose limit, as $h \to 0$, is $A'(t)$) . The numerator is our AREA from $x = a$ to $x = t+h$ MINUS the area from $x = a$ to $x = t$. Using properties of the definite integral we can rewrite this as: $\int_t^{t+h} f(x)\,dx$, the area from $x = t$ to $x = t+h$. The area of this narrow strip lies between the area of two rectangles, each of width "h": one has height equal to the minimum value of f(x) in [t,t+h] and the other has height equal to the maximum value of f(x) in [t,t+h]. This sounds reminiscent of the SQUEEZE theorem (!) which now comes to our rescue. We write:

$$h\, f(x)_{\text{minimum}} \le \int_t^{t+h} f(x)\,dx \le h\, f(x)_{\text{maximum}}$$

which, when divided through by h gives

$$f(x)_{\text{minimum}} \le \frac{1}{h}\int_t^{t+h} f(x)\,dx \le f(x)_{\text{maximum}}$$

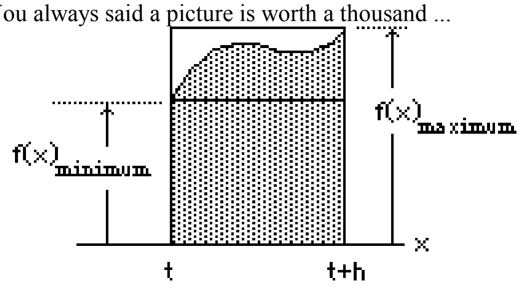
and now we <u>can</u> use the SQUEEZE theorem. Let h–>0 and recognize that the two "outside" limits are equal ... equal to what?

P: Did you hear? Equal to what?
S: I wasn't listening. Could you repeat the question?
P: Never mind. Think of the maximum value of f(x) in $t \le x \le t+h$. It occurs somewhere in that interval, say at $x = t_1$. Think also of the minimum value of f(x) in the same interval. Suppose it occurs at $x = t_2$. Then the above inequality reads : $f(t_1) \le \frac{1}{h}\int_t^{t+h} f(x)\,dx \le f(t_2)$. Okay, now let h–>0. What happens to t_1 and t_2?
S: I give up.
P: Think! They both lie in this interval from t to t+h, and that interval is shrinking to zero! They're both being squeezed to the left as h–>0. They're both approaching ... what?
S: I give up. I really wasn't listening. Besides, I haven't understood much of what you said ...
P: Pay attention! Both t_1 and t_2 are approaching x = t, the left-end of the interval. Then $f(t_1)$ and $f(t_2)$ are both approaching f(t) ... provided f(x) is continuous!! Remember I said we'll assume f(x) is continuous? Well, here's why. As h–>0 the left- and right-sides of the inequality $f(t_1) \le \frac{1}{h}\int_t^{t+h} f(x)\,dx \le f(t_2)$ have the same limit, namely f(t). Hence, the SQUEEZE theorem tell us that $\frac{1}{h}\int_t^{t+h} f(x)\,dx$ also has this limit. In other words

$$A'(t) = \lim_{h\to 0} \frac{A(t+h) - A(t)}{h} = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(x)\,dx = f(t)$$

and we've managed to differentiate this area function we've invented.
S: But this is confusing, isn't it? And you don't have any pictures! You always said a picture is worth a thousand ...
P: Okay, okay, here's a picture.

The shaded area is $A(t+h) - A(t) = \int_t^{t+h} f(x)\,dx$, the area under the curve from x = t to x = t+h. See? It's greater than the area of the rectangle whose height is the <u>minimum</u> value of f(x) and less than the area of the rectangle whose height is the <u>maximum</u> value of f(x). That means that A(t+h) - A(t) lies between h f_{min} and h f_{max}. That means that $\frac{A(t+h) - A(t)}{h}$ lies between f_{min} and f_{max}. Now, as h–>0, the minimum and maximum of f(x) (in the interval $t \le x \le t+h$) approach f(t). That makes the limit of



$h f(x)_{min} \le$ shaded area $\le h f(x)_{max}$.

$\frac{A(t+h) - A(t)}{h}$ equal to f(t) as well, because of the SQUEEZE theorem. That means we've managed to differentiate this rather weird function we've invented, and since $A(t) = \int_a^t f(x)\,dx$ gave us $A'(t) = f(t)$, we now see the general rule:

$\boxed{A'(t) = \text{the integrand evaluated at the upper limit}}$. In fact, if you remember the "differential", we would say $\frac{dA}{dt} = f(t)$ and replace $\frac{dA}{dt}$ by $\frac{\Delta A}{\Delta t}$ and get $\frac{\Delta A}{\Delta t} \approx f(t)$ so $\Delta A \approx f(t)\,\Delta t$ and we'd see that the tiny change in area is approximately $f(t)\,\Delta t$ which is the area of a certain rectangle. Nice, eh?

S: Okay, if you say so.

P: But we're not finished. Our goal is to evaluate $\int_a^b f(x)\,dx$ without having to resort to limits of Riemann sums. To see how the rest goes, let's do an example.

Example: Evaluate $\int_1^3 e^x\,dx$

Solution: First we introduce the "area function" $A(t) = \int_1^t e^x\,dx$. Remember that, eventually, we want A(3).

Now we differentiate A(t) and get: $A'(t) = e^t$, using the general rule, obtained from our investigation above: $\boxed{A'(t) = \text{the integrand evaluated at the upper limit}}$. Now that we know the derivative of A(t), can we find A(t) itself ... hence A(3)? Yes, because A(t) must be one of the functions $e^t + C$ (for some value of the constant C). These are the *only* functions which yield e^t when differentiated. But if $A(t) = e^t + C$, what's C? We know one other thing about $A(t) = \int_1^t e^x\,dx$ which we haven't used: $A(1) = \int_1^1 e^x\,dx = 0$, hence $A(1) = 0 = e^1 + C$ and we now know that $C = -e^1$. Finally, then, $A(t) = e^t - e^1$. When t = 3 we get $A(3) = e^3 - e$ (as we did before, when we used the Riemann sum definition of the definite integral).

The general procedure to evaluate $\int_a^b f(x)\,dx$ is as follows:

(1) Write $A(t) = \int_a^t f(x)\,dx$ (2) Then $A'(t) = f(t)$ (the integrand, evaluated at the upper limit).

(3) Then, if F(t) is <u>any</u> function whose derivative is f(t), we must have $A(t) = F(t) + C$.

(4) We also know that $A(a) = \int_a^a f(x)\,dx = 0$ and that gives $0 = F(a) + C$, hence $C = -F(a)$.

(5) Finally then, $A(t) = \int_a^t f(x)\,dx = F(t) - F(a)$ and, for t = b we get:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F(t) \text{ is any function whose derivative is } f(t).$$

Some Terminology:

If $\frac{d}{dx} F(x) = f(x)$, then we call F(x) an **ANTIDERIVATIVE** of f(x).

Examples:

- e^x is an antiderivative of e^x (because $\frac{d}{dx}\, e^x = e^x$).
- cos x is an antiderivative of - sin x (because $\frac{d}{dx}\cos x = -\sin x$)
- *ln* x is an antiderivative of $\frac{1}{x}$ (because $\frac{d}{dx}\ln x = \frac{1}{x}$) ... but only if x > 0.
- arctan x is an antiderivative of $\frac{1}{1+x^2}$ (because $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.
- *ln* sec x is an antiderivative of tan x (because $\frac{d}{dx}\ln \sec x = \frac{1}{\sec x}(\sec x \tan x) = \tan x$) ... but only if sec x>0.
- *ln* (sec x + tan x) is an antiderivative of sec x (check this out!) ... but only if sec x + tan x >0.

S: You keep saying "but only if *something* > 0". What's that all about?
P: Remember, the logarithm of (*something*) is defined only for positive *something* ... and *ln* is, of course, a logarithm.
S: Suppose you wanted an antiderivative of, say $\frac{1}{x}$, when x < 0?
P: Okay, let's consider $y = \frac{d}{dx}\ln(-x)$ because now -x <u>is</u> positive if x < 0. Let u = -x and get y = *ln* u so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}(-1)$ $= \frac{1}{-x}(-1) = \frac{1}{x}$. See? You still get $\frac{1}{x}$. That means that $\frac{d}{dx}\ln x = \frac{1}{x}$ when x > 0 and $\frac{d}{dx}\ln(-x) = \frac{1}{x}$ when x < 0. Nice, eh? We've got an antiderivative for $\frac{1}{x}$ for both positive and negative x. Of course we wouldn't expect an antiderivative which gives $\frac{1}{x}$ for x = 0 because $\frac{1}{x}$ doesn't even have a value at x = 0.
S: Does that mean we've gotta know in advance whether x is positive or negative ... else we won't know which to use? I mean, *ln* x or *ln* (-x) ... which one do we use?
P: What we need is a function, call him SAM, which is "x" when x > 0 and is "-x" when x < 0, then we can say that an antiderivative is *ln* SAM. Can you think of such a function?
S: You gotta be kiddin'. I'm sorry I asked.
P: How about | x |. Remember the absolute value function? It has precisely the right properties. Hence *ln* | x | is an antiderivative of $\frac{1}{x}$ for <u>all</u> $x \neq 0$.

Note: Every function has an infinite number of antiderivatives. If $\frac{d}{dx}F(x) = f(x)$, then $\frac{d}{dx}\big(F(x) + C\big) = f(x)$ as well so F(x) + C is also an antiderivative, regardless of the choice of constant C. To get the most general antiderivative, pick any one ... and add an arbitrary constant.

Examples:
$e^x + C$ is the most general antiderivative of e^x .
cos x + C is the most general antiderivative of - sin x .
ln | x | + C is the most general antiderivative of $\frac{1}{x}$ $(x \neq 0)$.
arctan x + C is the most general antiderivative of $\frac{1}{1+x^2}$.

Usually, antiderivatives are called INDEFINITE INTEGRALS and denoted by: $\int f(x)\,dx$ without any limits. In other words:

$$\boxed{\int f(x)\,dx \text{ denotes an ANTIDERIVATIVE (or INDEFINITE INTEGRAL) of } f(x)}$$

A few examples are in order (and can be easily verified by differentiating the right-side):

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\frac{x}{a} + C \qquad \int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Note: REMEMBER THE ABOVE INTEGRALS !!

Note: Remember that $\int \frac{dx}{a^2+x^2}$ is the $\int$um of elements and each has the dimensions of $\frac{dx}{a^2+x^2}$ which, if "a" and "x" are metres, has the dimensions of $\frac{1}{\text{metres}}$ so it's not surprising that this (indefinite) integral is $\frac{1}{a}$ arctan $\frac{x}{a}$ which also has this dimension (because of the factor $\frac{1}{a}$ ... not because of the arctan $\frac{x}{a}$ which is an angle in RADIANS hence has no dimensions). Note too that $\int \frac{dx}{\sqrt{a^2-x^2}}$ should be dimensionless ... which it is! These observations will help determine when the $\frac{1}{a}$ appears in the evaluation of the integral and when it doesn't.

The rule for evaluating DEFINITE integrals can now be restated as:

the FUNDAMENTAL THEOREM

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b \text{ where } F(x) \text{ is any antiderivative of } f(x)$$

and the notation $[F(x)]_a^b$ means F(b) - F(a)

Important observation: after having found an antiderivative F(x), then evaluating at the upper and lower limits and subtracting, the result is some number ... without any x's!

Examples: Evaluate the following definite integrals:

(a) $\int_0^{\pi} \sin x\,dx$ (b) $\int_0^{\pi/4} \frac{1}{1+x^2}\,dx$ (c) $\int_{-1}^{1} x^3\,dx$

Solutions:

(a) $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2$

Note, too, that $\int_0^{\pi} \sin t\,dt = [-\cos t]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2$ and $\int_0^{\pi} \sin z\,dz = [-\cos z]_0^{\pi} = 2$.

What name you give to the variable under the $\int$-sign is irrelevant.

S: Hold on! Why do you pick - cos x as the antiderivative. I thought you said that - cos x + C was the antiderivative of sin x? See? C! The constant of integration.
P: Right. Suppose I pick - cos x + C, then I'd evaluate it at the upper limit, at the lower limit, then subtract. I'd get $(-\cos\pi + C) - (-\cos 0 + C) = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2$. See? The C has cancelled out. Moral?
S: Forget the "C".
P: ... when evaluating <u>definite</u> integrals, but *never* when evaluating <u>indefinite</u> integrals!

(b) $$\int_0^1 \frac{1}{1+x^2}\,dx = [\arctan x]_0^1 = (\arctan 1) - \arctan 0 = \pi/4 - 0 = \pi/4$$

S: How do you know that arctan 1 = $\pi/4$?

P: If y = arctan x then x = tan y (because they're inverses, remember?). Hence, if we want to know y = arctan 1, then write this as 1 = tan y (everybody loves the tangent, nobody loves the arctangent, so we rewrite the relation to get it in terms of the tangent function)... so what's y? That is, what angle (in RADIANS!) has a tangent of 1? It's $\pi/4$, so that's y. See?

S: No. There are millions of angles whose tangent is 1. Am I right?

P: Already you've forgotten about our inverse trig functions! Remember! We had to restrict the domain so as to guarantee the existence of an inverse. That means that, if y = arctan x, then y must be chosen from the range $-\pi/2 < y < \pi/2$, and now there's only <u>one</u> angle there which has a tangent of 1 ... and it's $\pi/4$.

S: Okay, okay. I'm sorry I asked.

(c) $$\int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \left(\frac{1}{4}\right) - \left(\frac{1}{4}\right) = 0$$

The last is perhaps surprising. Does it *really* say that the area under $y = x^3$, from x = -1 to x = 1, is zero? We'll review a bit in the next lecture ... and see what this means.

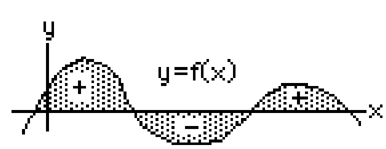
<u>LECTURE 15</u>

MORE ON DEFINITE INTEGRATION

We first considered the problem of finding the area enclosed by y = f(x), x = a, x = b and y = 0 (the x-axis), where f(x) was both continuous and positive in this interval. We then subdivided the area into the SUM of a multitude (namely "n") of tiny rectangles (because we have a formula for the area of rectangles). Each rectangle had a width $h = \frac{b-a}{n}$ and a height equal to some convenient value of f(x) in each subinterval. Letting the number of rectangles become infinite, we arrived at the AREA (the limit of the Riemann SUM) which we denoted by $\int_a^b f(x)\,dx$. Now, what happens if f(x) is always negative on [a,b]? We repeat the same procedure, but now the heights of the rectangles are all negative and we used (height)(width) to compute the area of each ... and (height) < 0 ... so we actually get the <u>negative</u> of the areas. The limit of the Riemann SUM as $n \to \infty$ yields not the area bounded by y = f(x), x = a, x = b and y = 0 but the <u>negative</u> of that area. (Note that we now avoid using the phrase "the area beneath y = f(x)" because this curve lies <u>below</u> the x-axis ... so the area below *it* is pretty large!)

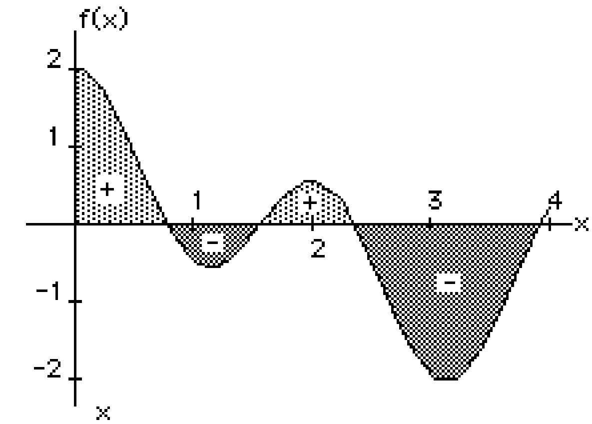
Now back to $\int_{-1}^{1} x^3\,dx$. We may break this up into two integrals (using one of the properties of definite integrals) and write: $\int_{-1}^{1} x^3\,dx = \int_{-1}^{0} x^3\,dx + \int_{0}^{1} x^3\,dx$ where the second integral is indeed the area "beneath" $y = x^3$ from x = 0 to x = 1 because $y = x^3$ is positive. However, on $-1 \le x \le 0$, $y = x^3$ is negative so the evaluation of this integral gives the <u>negative</u> of the area enclosed by the the curve, the x-axis and x = -1, x = 0. The integral actually gives $-\frac{1}{4}$, which cancels $\int_0^1 x^3\,dx = \frac{1}{4}$.

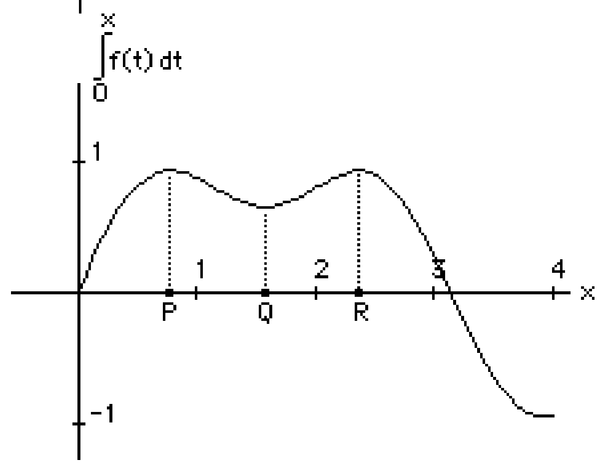
In other words, whenever y = f(x) lies below the x-axis, the definite integral yields the negative of the area so it's quite possible to have these "negative" areas cancel with "positive" areas to give zero.



Example: $\int_0^{2\pi} \sin x \, dx = [-\cos x]_0^{2\pi} = (-\cos 2\pi) - (\cos 0) = -1 + 1 = 0$ which seems clear since the "area" from 0 to π is positive while the "area" from π to 2π is negative ... and they cancel.

In the diagram on the right we plot the graph of some invented function f(x). Note that the "area" between the curve and the x-axis is sometimes positive, sometimes negative. We also plot the graph of $\int_0^x f(t)\,dt$ which is zero when x = 0, then increases until x = P (at x ≈ .8) because f(x) is positive so the "area" is increasing. After x = P, f(x) goes negative so the "area", namely $\int_0^x f(t)\,dt$, decreases, until x = Q (x ≈ 1.5) after which f(x) becomes positive again so the "area" increases again ... and so on. In fact, after x = R, (x ≈ 1.2) f(x) becomes so negative that the negative area cancels completely the positive area accumulated so far, and $\int_0^x f(t)\,dt$ goes negative.

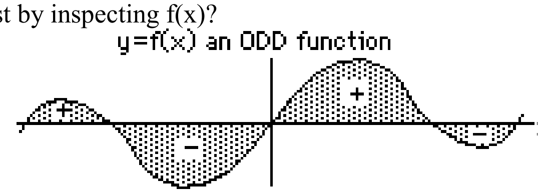




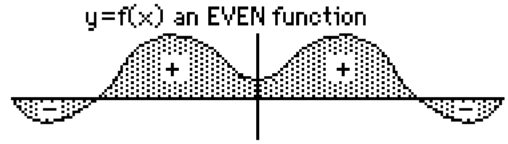
All this makes sense even if we didn't have any calculus or any fundamental theorem. However, we DO have the fundamental theorem and it says that $\frac{d}{dx} \int_0^x f(t)\,dt = f(x)$ so the derivative of the area function $A(x) = \int_0^x f(t)\,dt$ is actually f(x) and so it's clear that the graph of A(x) will have a horizontal tangent wherever f(x) = 0 ... and that's precisely what happens at P, Q and R!

Under what circumstances would you expect $\int_{-L}^{L} f(x)\,dx = 0$? Of course there must be "negative area" exactly cancelling "positive area" ... but can you predict that, just by inspecting f(x)?

Remember the definition of an ODD function? If you change the sign of x, you change the sign of f(x). That is, f(-x) = - f(x). Under these circumstances the values of the function on $-L \le x \le 0$ are just the negative of the values on $0 \le x \le L$ ... and the "areas" cancel.



How about $\int_{-L}^{L} f(x)\,dx$ when f(x) is an EVEN function? Now the values of f(x) on $-L \le x \le 0$ are exactly the same as those on $0 \le x \le L$ so the "areas" are as well. That is, $\int_{-L}^{0} f(x)\,dx = \int_{0}^{L} f(x)\,dx$. Hence we can write $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$ ... which is sometimes very convenient.



Examples: $\int_{-3}^{3} x \cos x\,dx = 0$ (because f(x) = x cos x is ODD) and

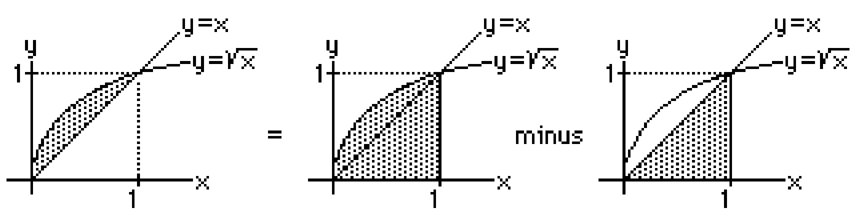
$\int_{-\pi}^{\pi} x^2 \sin x\,dx = 0$ (because $f(x) = x^2 \sin x$ is ODD) and

$\int_{-1}^{1} \sqrt{1 - x^2}\,dx = 2\int_{0}^{1} \sqrt{1 - x^2}\,dx = 2\left(\frac{1}{4}(\pi 1^2)\right)$ where we recognize $\int_{0}^{1} \sqrt{1 - x^2}\,dx$ as the area under $y = \sqrt{1 - x^2}$ from x = 0 to x = 1, namely the area of a quarter-circle of radius "1", i.e. $\frac{1}{4}\pi 1^2$. (Note that $y = \sqrt{1 - x^2}$ means that $x^2 + y^2 = 1$ ... so we *do* have a circle).

Let's calculate some areas:

Examples: Calculate the area (in the first quadrant) enclosed by the curves $y = \sqrt{x}$, y = x.

Solution: First we sketch the two functions so we can see the area:



Then we find the points of intersection ($y = \sqrt{x} = x$ has solutions: (0,0) and (1,1)).

Then we compute the area under $y = \sqrt{x}$ and subtract the area under y = x.

We get $\int_{0}^{1} x^{1/2}\,dx - \int_{0}^{1} x\,dx = [\frac{2}{3}x^{3/2}]_0^1 - [\frac{x^2}{2}]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

Finally, we check to see if this is reasonable! Note that the area lies within a square of area "1", so it should be <u>less</u> than this (which it is). In fact, it's clearly less than the area of half this square (since our area lies within the upper, triangular half) ... and it is. So we have some faith in our answer.

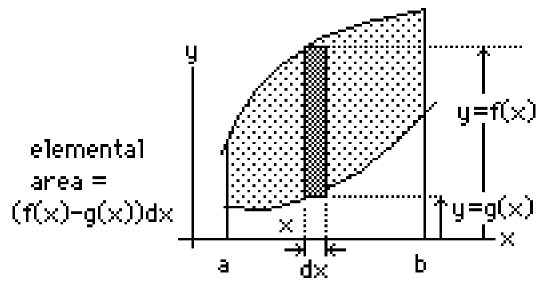
Note, however, that we could have used a property of definite integrals to combine the two integrals and write our area as:

$$\int_{0}^{1} (\sqrt{x} - x)\,dx = [\frac{2}{3}x^{3/2} - \frac{x^2}{2}]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} .$$

Since it's convenient to think of $\int_a^b f(x)\,dx$ as $\int$umming "elemental" rectangular areas of height f(x) and width dx (reflecting the origins of the definite integral as the limit of a Riemann SUM), then we can also think of $\int_0^1 (\sqrt{x} - x)\,dx$ as $\int$umming elemental areas of height $(\sqrt{x} - x)$ and width dx . We'll do this when convenient.

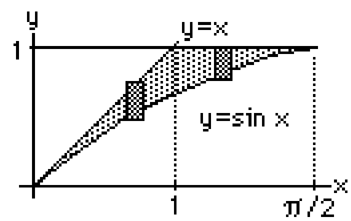
Indeed, if the required area is enclosed by curves y = f(x) and y = g(x) as shown, then we imagine subdividing the area into elemental rectangular areas located at the place x (somewhere between x = a and x = b), with width dx and height f(x) - g(x). The required area is the the $\int$um of these (beginning with rectangles at x = a and ending with x = b), yielding the definite integral:

$$\int_a^b (f(x) - g(x))\,dx$$



Example: Compute the first-quadrant area enclosed by y = x, y = 1 and y = sin x.
Solution: First we sketch the area.

Then we find the points of intersection: y = 1 = sin x at (π/2,1) and y = x = 1 at (1,1) and, of course y = x and y = sin x meet at the origin. We label these points on our sketch.



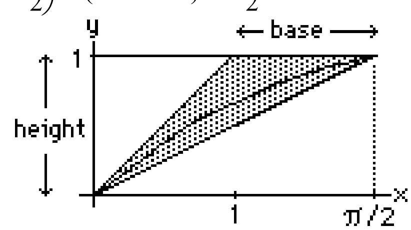
Then we imagine the area subdivided into elemental rectangular areas located at x, of width dx, and height ... well, that's a problem. When x lies in [0,1], the rectangle rises from y = sin x to y = x so has height (x - sin x) and area (x - sin x) dx. However, for x in [1,π/2] the rectangle rises from y = sin x and ends on y = 1 hence has area (1 - sin x) dx. We'll need to find the areas separately: $\int_0^1 (x - \sin x)\,dx + \int_1^{\pi/2} (1 - \sin x)\,dx$

$= \left[\frac{x^2}{2} + \cos x\right]_0^1 + \left[x + \cos x\right]_1^{\pi/2} = \left(\frac{1}{2} + \cos 1\right) - (0 + \cos 0) + \left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (1 + \cos 1) = \frac{\pi - 3}{2}$.

Finally, we check for *reasonableness*. Our area is certainly less than the area of the triangle shown and this triangle has area

(1/2) (base) (height) = (1/2) (π/2 - 1) (1) = $\frac{\pi/2 - 1}{2}$.

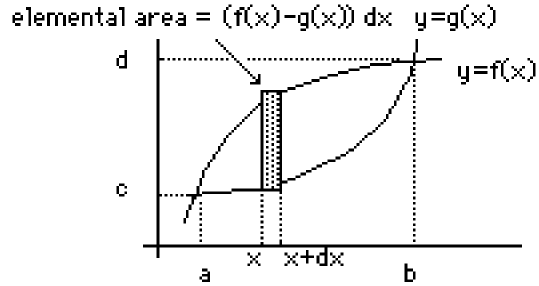
Is $\frac{\pi - 3}{2} < \frac{\pi/2 - 1}{2}$? Yes, so we have some faith in our answer.



S: Some faith? What does *that* mean?
P: You'd be surprised how many students will get an answer like 12,473 then go on to the next question, thinking they're finished.
S: Really? You're kidding. I'd never do that.
P: We'll see.

We've seen that the definite integral (which gives the required area) can be generated by imagining a subdivision into elemental rectangles, determining the area of these rectangles, then $\int$umming from the first rectangle (at x = a, say) to the last (at x = b, say).

Let's review this technique:



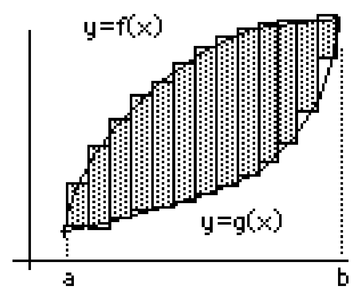
Suppose we want the area enclosed by y = f(x) and y = g(x). We first sketch the area, then find the points of intersection by solving y = f(x) and y = g(x) (<u>two</u> equations in <u>two</u> unknowns). Suppose the intersections are at (a,c) and (b,d), so the area we seek lies between x = a and x = b, y = c and y = d, as shown. Then we subdivide into elemental rectangular areas located at the place x, of width dx and height f(x) - g(x) (the upper y-value *minus* the lower y-value), hence the area (f(x) - g(x)) dx. Now we $\int$um these rectangular areas from x = a to x = b via the definite integral: $\int_a^b (f(x) - g(x))\, dx$.

S: That's not right, you know. I mean, the height isn't really f(x) - g(x). That's only the height at x, not at x + dx. So how come it gives the right answer ... or does it?

P: Remember that we got $\int_a^b (f(x) - g(x))\, dx$ by taking the limit of a Riemann SUM ... <u>not</u> by this gimmick of imagining elemental rectangles and $\int$umming . We do this "elemental rectangle" business in order to get the definite integral which represents the area <u>without</u> always having to go back and construct a Riemann SUM ... but I can if you'd like.

S: Sure. Go ahead. I've got lots of time.

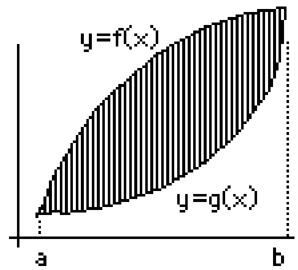
P: Okay, we subdivide [a,b] into n subintervals of width $h = \frac{b-a}{n}$. The x-coordinates of the points of subdivision are x=a, a+h, a+2h, a+3h, ..., a+nh (which is also "b"). On each subinterval we pick a convenient x-value and compute f(x) - g(x) at that value ... and construct a rectangle with *this* height, and width h. For example, suppose that we choose x_1 in the first subinterval, x_2 in the second and so on. Then $f(x_1) - g(x_1)$ is the height of the first rectangle, $f(x_2) - g(x_2)$ the height of the second, etc. Now we construct a Riemann SUM which is the sum of the areas of all n rectangles:



$\big(f(x_1) - g(x_1)\big)\, h + \big(f(x_2) - g(x_2)\big)\, h + ...+ \big(f(x_n) - g(x_n)\big)\, h = \sum_{k=1}^{n} \big(f(x_k) - g(x_k)\big) h$ (in sigma notation) . Finally we let $n \to \infty$ (so that $h = \frac{b-a}{n} \to 0$ as well) and <u>define</u> the limit as $\int_a^b (f(x) - g(x))\, dx$. See? That's the area we're after ... but we could have gotten to this definite integral by imagining a bunch of "elemental" rectangles of area $\big(f(x)-g(x)\big)\, dx$ and $\int$umming from x = a to x = b.

S: Actually, the area of your rectangles don't look much like the area we're after. I guess the definite integral is only a good approximation, right?

P: No, it's <u>exact</u>. I just chose a few subdivisions in my diagram so you can actually *see* the individual rectangles. Here, let me show you what it looks like with 47,000,000 elemental rectangles. See? The elemental rectangles really give a good approximation ..

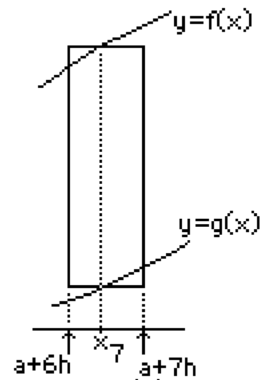


S: Aha! An approximation! Didn't I just say ... ?

P: Wait, it's only an approximation if you take 100 or 47,000,000 rectangles. The error gets smaller as the number of rectangles goes up. In the limit, as $n \to \infty$, the error goes to zero ... and the definite integral is <u>exactly</u> the area. Anyway, what's your favourite number?

S: Huh? ... seven, I guess.

P: Then let's look carefully at the 7^{th} subinterval which goes from $x = a + 6h$ to $x = a + 7h$ and we'll pick a convenient x-value in this subinterval and called it x_7 and construct a rectangle with height $f(x_7) - g(x_7)$ and width h. It doesn't look like a very good approximation to the actual area between f(x) and g(x), but as $h \to 0$ and this rectangle becomes skinny and the number of such rectangles becomes infinite ... voila! The SUM of all of them has the correct limit!

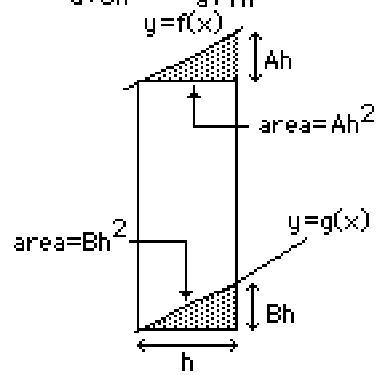


S: That's amazing, isn't it? I mean, all those errors ... millions of them ... and you add them up and they go to zero. You'd think that more rectangles would give more errors and the approximation would get worse and worse and ..

P: How big do you think each error is?

S: I haven't the foggiest idea.

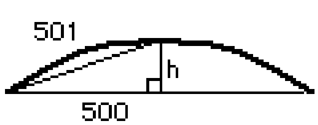
P: Let's estimate. If f(x) and g(x) were constants, then there'd be <u>no</u> error, but if f(x) is increasing at a maximum rate A (meaning y = f(x) is increasing A times more rapidly than x), then y changes by about Ah when x changes by an amount h. Similarly, if g(x) is changing at the rate B (that's g'(x), by the way) then y = g(x) changes by an amount Bh, roughly. The error in area, for each rectangle, is no bigger than the sum of the shaded areas, and each is roughly a triangle so we can estimate using (1/2) (base)(height) and get $(1/2)(h)(Ah) = (A/2)h^2$ and $(1/2)(h)(Bh) = (B/2)h^2$ which adds up to $(A/2+B/2)h^2$ and putting $h=(b-a)/n$ we get $(A/2+B/2)(b-a)^2/n^2$. That means that the error in each triangle is proportional to $1/n^2$ and although we have n such triangles we'd get a total error proportional to 1/n which <u>does</u> go to zero as $n \to \infty$. Nice, eh?



S: Triangles? Did you say the shaded areas were triangles? But ...

P: No, no ... they're *approximately* triangles. It's like assuming that f(x) and g(x) were replaced (for purposes of this estimation) by their linear, tangent line approximation ... surely you remember the linear approximation? Besides, triangles sometimes come in handy for estimating. Here, let me give you a problem you can't solve but which you can estimate, using triangles. Ready? There is this train track and it's one long piece of steel 1,000 metres long and it's pinned down at each end so the ends can't move. Then it gets hot and the increased temperature expands the metal by 2 metres and the track buckles in the centre. Got the picture?

S: A picture is worth ..



P: Okay, here's the picture. My question is: how much does it buckle in the centre? If we call it "h" then we can estimate this using a triangle. See the triangle? We have $h^2 + 500^2 = 501^2$ (using Pythagorus) so $h^2 = 501^2 - 500^2 = 1001$ so $h = \sqrt{1001} \approx$ ... uh, did you bring your calculator?

S: Nope.

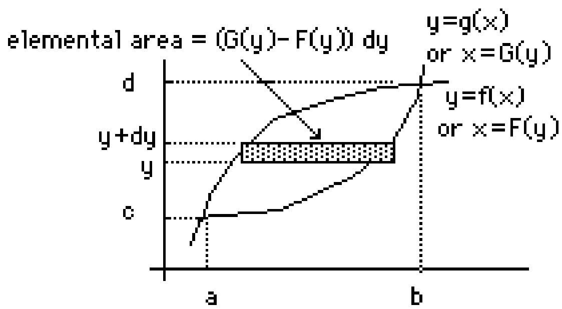
P: Okay, we'll do a linear approximation on this one. Let's see ... 1001 is pretty close to 900 which is 30^2, but 31^2 is 961 so I'll use it and get the tangent line to $\sqrt{x}$ at x = 961. That's $f(961) + f'(961)(x - 961)$ and for $f(x) = x^{1/2}$, we have $f'(x) = 1/2\sqrt{x}$ so $f(961) + f'(961)(x - 961) = \sqrt{961} + \frac{1}{2\sqrt{961}}(x - 961) = 31 + \frac{x-961}{62}$ so $\sqrt{x} \approx 31 + \frac{x-961}{62}$ when x is close to 961. I plug in x = 1001 and get $\sqrt{1001} \approx 31 + \frac{40}{61} \approx 31.6$ which is good enough for me.

S: For me too. (I hope he doesn't mean 31.6 metres!?) Say, do I have to know this for the final exam?

P: Train tracks? No. Linear approximation? Yes. Now pay attention, because if you're satisfied that this "elemental rectangle" gimmick gives the correct definite integral, then I'll show you something really neat.

S: I can hardly wait.

We consider the same problem as above: the area enclosed by y = f(x) and y = g(x). Rather than subdividing the area into vertical "elemental rectangles" (located at x, between x = a and x = b), we subdivide into horizontal "elemental rectangles" ... located at y between y = c and y = d.



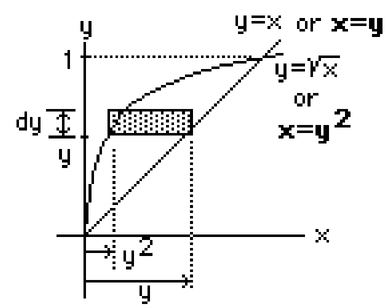
To find the area of this horizontal rectangle we use (width) (height) where the height is just dy and the width is the "right x-value" *minus* the "left x-value". That means we need to solve y = f(x) for x in terms of y ... and y = g(x) for x in terms of y. Suppose this gives x = F(y) and x = G(y) respectively. Then, for the rectangle located at "y", the width is G(y) - F(y) so the area of this "elemental rectangle" is (G(y) - F(y)) dy and $\int$umming all such rectangles, from y = c to y = d

gives the definite integral: $\boxed{\int_c^d (G(y) - F(y))\, dy}$

Example: Calculate the area enclosed by y = x and y = $\sqrt{x}$ using horizontal rectangles.

Solution: We first sketch the curves and identify the area. Then we find the points of intersection by solving y = x and y = $\sqrt{x}$ (and get (0,0) and (1,1)) . Then we imagine a single, typical horizontal elemental rectangle located at y, with height dy, and compute its area. For a given y we need to know the *right x-value* and the *left x-value* which means we need to solve y = x and y = $\sqrt{x}$ for x in terms of y; that's easy. **x = y** and **x = y²** → **x = y^2**. Hence our horizontal rectangle has area (y - y^2) dy and we $\int$um all such areas from y = 0 to y = 1 and get $\int_0^1 (y - y^2)\, dy$

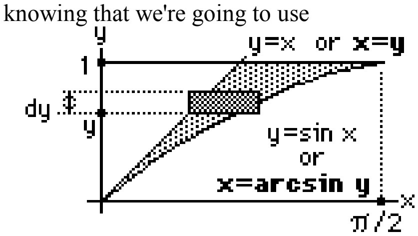
$= [\frac{y^2}{2} - \frac{y^3}{3}]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ and (surprise!) it's the same area as we computed using vertical rectangles.



S: So why use horizontal rectangles at all?

P: I knew you'd ask that. Sometimes it's easier. Let's go back to the area enclosed by y = x, y = 1 and y = sin x. We sketch the curves, find the points of intersection (we've already done that, earlier) and, knowing that we're going to use horizontal rectangles we immediately solve for x in terms of y. It's fortunate that we've already considered the arcsine function because y = sin x, solved for x, gives x = arcsin y (the inverse function, remember?). Okay, now we sketch a typical horizontal rectangle located at "y", between y = 0 and y = 1. Its area is:

$(x_{right} - x_{left})$ dy = (arcsin y - y) dy and $\int$umming all such rectangles



gives the definite integral: $\boxed{\int_0^1 (\arcsin y - y)\, dy}$. Nice, eh?

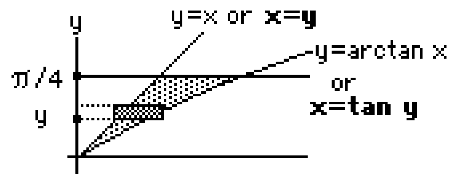
If you look back at this problem when we used vertical rectangles, we had to divide the x-interval into two pieces: $0 \le x \le 1$ and $1 \le x \le \pi/2$ because the expression for elemental area changed. But horizontal rectangles <u>always</u> go from the line x = y to the curve x = arcsin y, so there's one integral to evaluate instead of two!

S: But how would you find the antiderivative of arcsin y? I mean, what's $\int_0^1 \arcsin y \, dy$?

P: Uh ... good question. You'll have to wait till later.
S: Thanks.

Example: Find the area enclosed by y = arctan x, $y = \frac{\pi}{4}$ and the tangent line to y = arctan x at x = 0.

Solution: The tangent line has slope $\frac{dy}{dx} = \frac{d}{dx} \arctan x = \frac{1}{1+x^2} = 1$, at x = 0 and y = arctan 0 = 0 (when x = 0) so it's $\frac{y - 0}{x - 0} = 1$ or simply y = x. Now we sketch the curves y = arctan x, $y = \frac{\pi}{4}$ and y = x, identify the area, and find
the points of intersection: y = x intersects y = π/4 at (π/4,π/4) and y = arctan x intersects y = π/4 at (1,π/4) and, of course, y = x and y = arctan x intersect at (0,0). Now we draw a typical horizontal rectangle located at "y", between y = 0 and y = π/4. Its height is dy and its width is $(x_{right} - x_{left})$ so its area is:



(tan y - y) dy and $\int$umming gives:

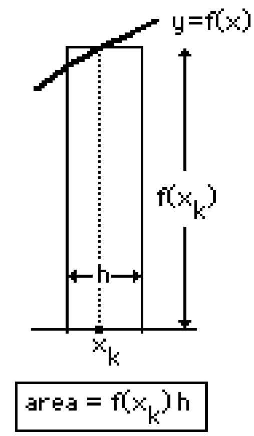
$$\int_0^{\pi/4} (\tan y - y)\, dy = \left[\ln|\sec y| - \frac{y^2}{2} \right]_0^{\pi/4} = \left(\ln|\sec \pi/4| - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 \right) - \left(\ln|\sec 0| - 0 \right) = \ln\sqrt{2} - \frac{\pi^2}{32} .$$

S: Hah! You didn't check for reasonableness!
P: Okay. Checking for reasonableness, we compare to the area of the triangle (which looks much like the shaded region in the diagram): (1/2) (base) (height) = (1/2) (1 - π/4) (π/4) which is roughly (1/2) (1 - 3/4)(3/4) = 3/32 ≈ .1 whereas $\ln\sqrt{2} - \frac{\pi^2}{32}$ is roughly ... uh ... well ...
S: Gotcha!
P: I'll use the linear, tangent line approximation to estimate $\ln\sqrt{2}$. Let's see ... $\sqrt{2}$ is near 1, so I'll find the tangent line to y = *ln* x at x = 1 and that has a slope of $\frac{d}{dx} \ln x = \frac{1}{x} = 1$ (at x = 1) so the line is $\frac{y - \ln 1}{x - 1} = 1$ and that's y = x - 1 (since I know that *ln* 1 = 0) and putting $x = \sqrt{2}$, I estimate $\ln\sqrt{2} \approx \sqrt{2} - 1$ and I know that $\sqrt{2}$ is roughly 1.4 so I get $\ln\sqrt{2} \approx .4$ and subtracting $\frac{\pi^2}{32} \approx \frac{10}{32} \approx .3$, I get .1 which checks out and now you're really impressed, right?
S: zzzzz ... uh, say ... you seem to be fond of approximations. Everything is an approximation. Isn't anything exact in math?
P: Sure, but what would you do with the <u>exact</u> value of $\sqrt{2}$ for example. As a decimal it has an infinite number of digits. Do you really want to see them all? We can use Newton's method to get, say, a million digits and that should be good enough. Besides, sometimes approximate answers can give exact results when you use limits. We've already seen that haven't we? From the Riemann SUM, which is an approximation, we take a limit and get the area under the curve <u>exactly</u>.
S: Can't we just go on?

LECTURE 16

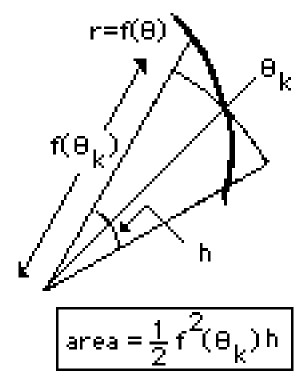
AREAS IN POLAR COORDINATES

In order to compute the area enclosed by curves in rectangular coordinates ($y = \sqrt{x}$, $y = \arcsin x$, $x = a$, $x = \pi/4$ and curves like that) we subdivide into elemental rectangles and $\int$um using the definite integral. In order to compute the area enclosed by curves in polar coordinates ($r = \cos\theta$, $r = \theta$, $r = 3 + 4\sin\theta$ and curves like that) we do a similar thing ... except that rectangular "elemental areas" are now inappropriate. To see what *is* appropriate, let's repeat the prescription for finding the area "under" $y = f(x)$ (i.e. from $y = 0$ to $y = f(x)$) from $x = a$ to $x = b$ ... except we'll word it slightly differently so we can repeat it for polar coordinates:



(1) Subdivide $a \le x \le b$ into n subintervals of length $h = \dfrac{b - a}{n}$.
(2) On each subinterval pick a convenient value of x, say x_k on the k^{th} subinterval, and assume $y = f(x)$ is constant at the value $f(x_k)$ over the k^{th} subinterval.
(3) Determine the area from $y = 0$ to $y = f(x_k)$, on the k^{th} subinterval.
(4) Sum all such areas to get a Riemann SUM.
(5) Take the limit at $n \to \infty$ and hence $h \to 0$.
(6) This limit (assuming there *is* a limit) is the required AREA.

Now, in polar coordinates:



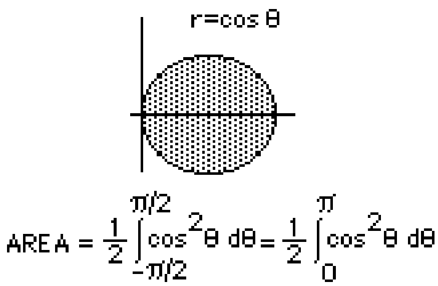
(1) Subdivide $a \le \theta \le b$ into n subintervals of length $h = \dfrac{b - a}{n}$.
(2) On each subinterval pick a convenient value of θ, say θ_k on the k^{th} subinterval, and assume $r = f(\theta)$ is constant at the value $f(\theta_k)$ over the k^{th} subinterval.
(3) Determine the area from $r = 0$ to $r = f(\theta_k)$, on the k^{th} subinterval.
(4) Sum all such areas to get a Riemann SUM.
(5) Take the limit at $n \to \infty$ and hence $h \to 0$.
(6) This limit (assuming there *is* a limit) is the required AREA.

The difference is in step (3) where the "elemental area" is a sector of a circle rather than a rectangle (and this seems appropriate for polar coordinates, right?). To deduce this area (in case you haven't memorized such) we recall the area of a circle of radius r: πr^2. If we take just a fraction of this area, say the fraction $\dfrac{h}{2\pi}$, then we get the area of a sector where the central angle is "h" (instead of "2π"). Hence the sector has area $\left(\dfrac{h}{2\pi}\right) \pi r^2 = \dfrac{1}{2}\, r^2 h$ and that's what we use for the elemental area. The Riemann SUM would be $\displaystyle\sum_{k=1}^{n} \frac{1}{2} f^2(\theta_k)\, h$ and its limit would become the definite integral: $\boxed{\displaystyle\int_a^b \frac{1}{2} f^2(\theta)\, d\theta}$ rather than $\displaystyle\int_a^b f(x)\, dx$.

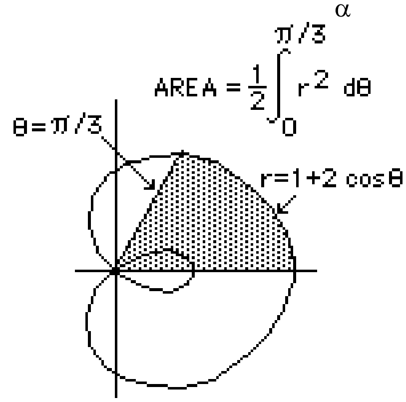
> The area enclosed by $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ is given by: $\frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\, d\theta$

S: So how come you switched from a, b to α, β?
P: Everybody uses Greek letters for angles ... didn't you know that? Pay attention:

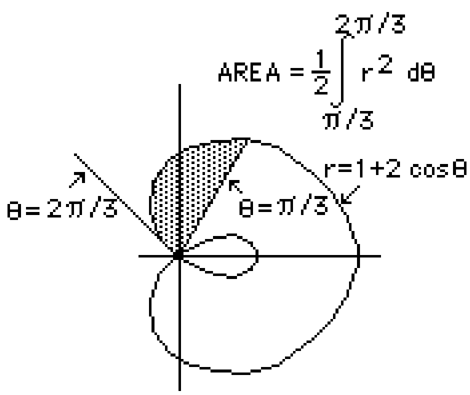
It's important to regard $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$ as the <u>AREA SWEPT OUT BY THE RADIUS</u> $r = f(\theta)$ as θ goes from $\theta = \alpha$ to $\theta = \beta$. To emphasize this point we'll sketch some polar curves and shade the area which $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$ gives:



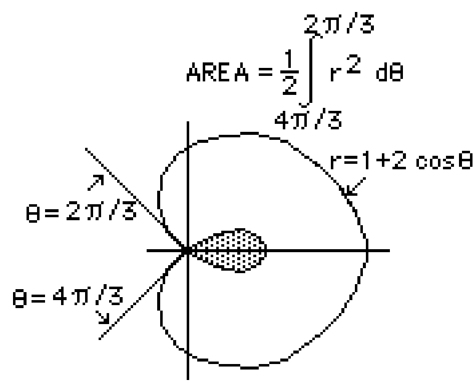
Here, the polar curve $r = \cos\theta$ is traversed once when θ goes from 0 to $\frac{\pi}{2}$ ... or from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$



Here the integral
Error!



Here the limaçon reaches $r = 0$ when $\theta = \frac{2\pi}{3}$ so $\frac{1}{2}\int_{\pi/3}^{2\pi/3} r^2\, d\theta$ gives the area shown.

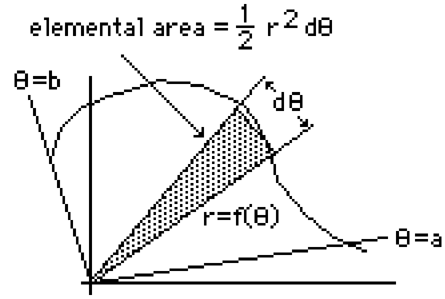


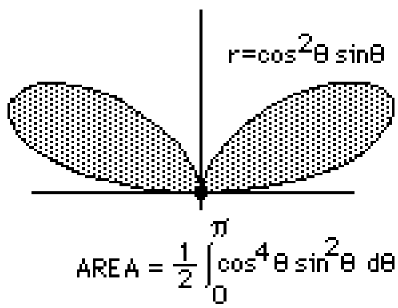
Here r traces out the inner loop as θ goes from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ so $\frac{1}{2}\int_{2\pi/3}^{4\pi/3} r^2\, d\theta$ gives the area of this inner loop.

It's important to note that, for example, $\frac{1}{2}\int_{0}^{2\pi} \cos^2\theta\, d\theta$ would give <u>twice</u> the area of the circle and

$\frac{1}{2}\int_{0}^{2\pi}(1+2\cos\theta)^2 d\theta$ will give the area within the limaçon with the inner loop counted twice!

Remember the diagram ===>>>





Example: Express, as a definite integral, the area enclosed by the curve $r = \cos^2\theta\sin\theta$.

Solution: We sketch the curve (a lemniscate ... we've done this one before ...) and observe that the curve is traversed in its entirety when θ goes from 0 to π (and no more!). The area swept out is given in the diagram.

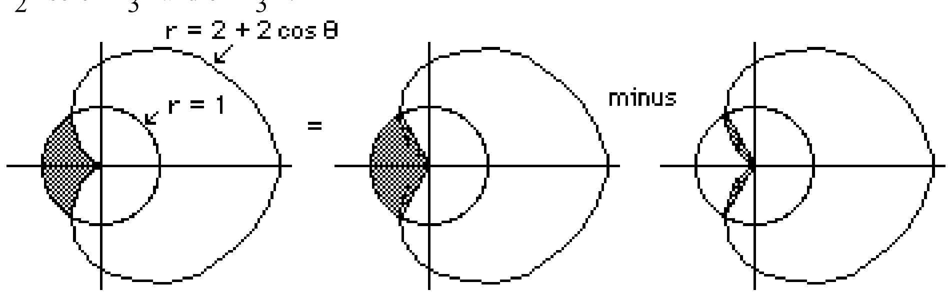
S: I'm not sure I get this "swept out" business. Sounds like a broom or ...

P: Pay attention: you imagine the radius r as a straight line from the origin to a point on r = f(θ). Then, as θ changes, this straight line sweeps out an area. That can't be hard to understand. In fact, one of Kepler's three laws of planetary motion says that the area swept out is the same for any given time period. Here the origin is the sun and a planet is revolving about the sun along some path r = f(θ) and each month (or day or year) the area swept out is the same and that means that when the planet is close to the sun and r is small it must be travelling quickly whereas when r is large it travels slowly so as to keep the area swept out the same for ... are you listening?

S: zzzz

Example: Find the area *outside* r = 2 + 2 cos θ but *inside* r = 1.

Solution: We sketch both polar curves and find the points of intersection by solving r = 2 + 2 cos θ = 1. This gives $\cos\theta = -\frac{1}{2}$ so $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.



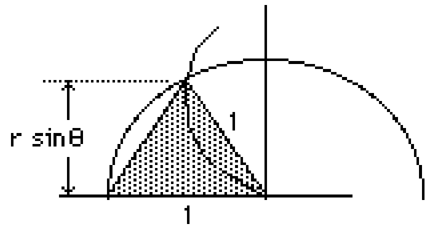
Now the integral $\frac{1}{2}\int_{2\pi/3}^{4\pi/3}(2+2\cos\theta)^2 d\theta$ will give the area swept out as "r" traces the limaçon while

$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} 1^2\, d\theta$ gives the area swept out when r traces out the circle. We want the difference:

$$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} 1^2\,d\theta\ -\frac{1}{2}\int_{2\pi/3}^{4\pi/3}(2+2\cos\theta)^2 d\theta\ = -\frac{1}{2}\int_{2\pi/3}^{4\pi/3}\big(3+8\cos\theta+4\cos^2\theta\big)\,d\theta\ .\ \text{Put } \cos^2\theta = \frac{1+\cos2\theta}{2}\ \text{ and get}$$

$$-\frac{1}{2}\int_{2\pi/3}^{4\pi/3}(5+8\cos\theta+2\cos2\theta)\,d\theta\ = -\frac{1}{2}\ \Big[5\theta+8\sin\theta+\sin2\theta\Big]_{2\pi/3}^{4\pi/3}\ = -\frac{1}{2}\left(\frac{20\pi}{3}+8\left(-\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(\frac{10\pi}{3}+8\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)\ = \frac{7}{2}\sqrt{3}\ -\frac{5\pi}{3}$$

S: Reasonableness! Reasonableness!
P: <u>You</u> check it for reasonableness.
S: Huh? Well ... let's see ... are there any triangles around?
How about this one =====>>> ?
The base is 1 and the height is ... $y = r\sin\theta$ and that's ...
uh, (1) $\sin(2\pi/3) = \sin 120^\circ = \sqrt{3}/2$ so the area is
$\frac{1}{2}(1)(\sqrt{3}/2) = \sqrt{3}/4$, right? And $\sqrt{3}$ is maybe 1.5 or
something so I get .4 or something. Good?



P: How does that compare with the exact answer?
S: The exact answer is $3.5\sqrt{3} - 5\pi/3$ and that's about (3.5)(1.5) - (5)(3)/3 = 5.25 - 5 = .25 which is way off ... so your answer is wrong!
P: You said $\sqrt{3}$ is *maybe* 1.5, but it's about 1.73 so that'd change your $\sqrt{3}/4$ estimate to about .43 and the exact answer to about .86 and ...
S: See? You <u>are</u> wrong!
P: ... and you only estimated the upper half of the area so if I double your estimate I'd get ...
S: Hey, it's .86, right on the button! See?
P: Hmmm. Now if you only had me around when you write your final exam.

<u>LECTURE 17</u>

TECHNIQUES OF INTEGRATION

We collect those "indefinite" integrals which we already know (and which you should remember!):

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$	$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C\ (a \neq 0)$	$\int \frac{dx}{x} = \mathit{ln}\,\lvert x \rvert + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$	$\int \tan x\,dx = \mathit{ln}\,\lvert \sec x \rvert + C$
$\int \sec x\,dx = \mathit{ln}\,\lvert \sec x + \tan x \rvert + C$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

We could add to this list by differentiating a great number of functions then we'd know the antiderivatives of the answers. i.e. since $\frac{d}{dx}(x\,\mathit{ln}\,\lvert\cos x\rvert) = x\left(\frac{1}{\cos x}(-\sin x)\right) + \mathit{ln}\,\lvert\cos x\rvert = -x\tan x + \mathit{ln}\,\lvert\cos x\rvert$ then $\int(-x\tan x + \mathit{ln}\,\lvert\cos x\rvert)dx = x\,\mathit{ln}\,\lvert\cos x\rvert + C$. However, we need a more systematic way of evaluating integrals (and we'll say "integrals" and mean "indefinite integrals" unless otherwise indicated).

Note, however, that every formula for differentiation will yield a formula for integration (and, again, we mean "indefinite" integration unless otherwise indicated). For example we have $\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$ so we have the integration formula: $\int (f'(x) + g'(x))\,dx = f(x) + g(x) + C$ ... not too exciting.

THE METHOD OF SUBSTITUTION:

As a "Differentiation Rule" we also have the Chain Rule: $\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$ where u is some (differentiable!) function of x. This gives the "Integration Rule": $\int \left(f'(u)\frac{du}{dx}\right) dx = f(u) + C$, which, if we could understand what it's saying, looks pretty interesting.

Note: if we replace $\frac{du}{dx}$ dx by du (as though we were "cancelling the dx", the above result reads:

$\int f'(u)\,du = f(u) + C$ which is pretty obvious (i.e. f(u) is clearly the antiderivative of f'(u) !) ... so maybe the Chain Rule is of little value, but then again, this rule works for <u>any</u> differentiable functions f & u so maybe there's something to it ... so we should try a few f's and u's.

- If f(u) = sin u then $\int \cos u \frac{du}{dx}\,dx = \sin u$, for any (differentiable) function "u". If $u = x^2$, we get:

  $\boxed{\int \cos x^2\, 2x\,dx = \sin x^2 + C}$.

- If $f(u) = e^u$ and u = sin x we get the result $\boxed{\int e^{\sin x} \cos x\,dx = e^{\sin x} + C}$.

- How to identify the function "u", given an integral to evaluate: $\int F(x)\,dx$?

  Look for a function of x whose derivative is "next to dx" (i.e. is a factor of integrand), and let *u = this function*!!

  <u>That's</u> the method of substitution we'll talk about:

- $I = \int \cos x^2\, 2x\,dx$ = ? Let $u = x^2$ (since its derivative, 2x, is "next to dx"), so $\frac{du}{dx} = 2x$ hence $du = \frac{du}{dx}\,dx =$ 2x dx. Now write $I = \int \cos u \frac{du}{dx}\,dx = \int \cos u\,du = \sin u + C = \sin x^2 + C$ (as obtained above). Note: normally the indefinite integral would most likely arise as $\int x \cos x^2\,dx$ (without the "2" and without the x being "next to dx") , so you could rewrite it as $\frac{1}{2}\int \cos x^2\, 2x\,dx$, or simply ignore constant multipliers and set $u = x^2$, $du = \frac{du}{dx}\,dx = 2x\,dx$ so $\frac{1}{2}\,du = x\,dx$ giving $\int x \cos x^2\,dx = \frac{1}{2}\int \cos u\,du$ etc.

- $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ = ? Note that $u = \sqrt{x}$ has derivative $\frac{1}{2\sqrt{x}}$ which (except for a constant factor "2") can be placed "next to dx" (i.e. is a factor of the integrand). Then put $u = \sqrt{x}$ and $du = \frac{du}{dx}\,dx = \frac{1}{2\sqrt{x}}\,dx$

so put $\frac{1}{\sqrt{x}}\,dx = 2\,du$, giving $I = \int \sin u\; 2\,du = -2\cos u + C = -2\cos\sqrt{x} + C$.

- $\int \frac{\ln x}{x}\,dx = ?$ Let $u = \ln x$ (its derivative, $\frac{du}{dx} = \frac{1}{x}$, is a factor of the integrand) and $du = \frac{du}{dx}\,dx = \frac{1}{x}\,dx$ and reduce this tough integral to simply $\int u\,du = \frac{u^2}{2} + C = \frac{\mathit{ln}^2 x}{2} + C$.

- $\int \frac{x}{1+x^2}\,dx = ?$ Let $u = 1+x^2$ (its derivative, $2x$, is a factor of the integrand ... except for the "2" which we ignore ... it'll appear, soon) then $du = \frac{du}{dx}\,dx = 2x\,dx$ hence $\frac{1}{2}\,du = x\,dx$ and get: $\int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\,\mathit{ln}\,|u| + C$
$= \frac{1}{2}\,\mathit{ln}\,(1+x^2) + C$ and we conclude that $\int \frac{x}{1+x^2}\,dx = = \frac{1}{2}\,\mathit{ln}\,(1+x^2) + C$ ... which we can verify by noting that $\frac{d}{dx}\left(\frac{1}{2}\,\mathit{ln}\,(1+x^2) + C\right) = \frac{x}{1+x^2}$.

- $\int \frac{x}{\sqrt{x^2+1}}\,dx = ?$ Let $u = x^2+1$ (its derivative, $\frac{du}{dx} = 2x$, is a factor of the integrand ... except for the "2" which we ignore ... it'll appear, soon) then $du = \frac{du}{dx}\,dx = 2x\,dx$ hence $\frac{1}{2}\,du = x\,dx$ and get: $\int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du$
$= \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = u^{1/2} + C = \sqrt{x^2+1} + C$.

- $\int e^x \tan(e^x)\,dx = ?$ Let $u = e^x$ (its derivative, $\frac{du}{dx} = e^x$, is a factor of the integrand) and get $\int \tan u\,du$
$= \int \frac{\sin u}{\cos u}\,du$ so <u>now</u> let $v = \cos u$ (<u>its</u> derivative, $\frac{dv}{du} = -\sin u$, is a factor of the integrand) and get $\int \frac{-\,dv}{v} = -$
$\mathit{ln}\,|\,v\,| + C = -\,\mathit{ln}\,|\cos u\,| + C = -\,\mathit{ln}\,|\cos e^x\,| + C$ (which, of course, we should verify by differentiation).

- $\int \sec^2(\sin x)\cos x\,dx = ?$ Let $u = \sin x$ so $du = \frac{du}{dx}\,dx = \cos x\,dx$ and we get $\int \sec^2 u\,du = \tan u + C$
$= \tan(\sin x) + C$.
- $\int \frac{1}{\mathit{ln}\,x}\,\frac{1}{x}\,dx = ?$ Let $u = \mathit{ln}\,x$ then $du = \frac{du}{dx}\,dx = \frac{1}{x}\,dx$ and we get $\int \frac{1}{u}\,du = \mathit{ln}\,|\,u\,| + C = \mathit{ln}\,\,|\mathit{ln}\,x\,| + C$.
- $\int e^{\sin x^2}\,2x\sin x^2\,dx = ?$ Let $u = \sin x^2$ then $du = \frac{du}{dx}\,dx = 2x\sin x^2\,dx$ and we get $\int e^u\,du = e^u + C$
$= e^{\sin x^2} + C$.

If we inspect the integrals that we've been able to evaluate with this "rule", then we see that there is always some function appearing in the integrand and the derivative of this function also appears there, right beside the "dx".

If you <u>can</u> find a function in the integrand (call it g(x)) whose derivative is also in the integrand "next to dx" (that is, $\frac{dg}{dx}$ dx is <u>also</u> there, under the ∫-sign) i.e. the integral has the form

$\boxed{\int \{\text{some hairy function involving } g(x)\}\ g'(x)\,dx}$ then let u = g(x) and replace g(x) wherever it occurs in the integrand by u (which will certainly make for a simpler integrand!) AND replace $\frac{du}{dx}\,dx = \frac{dg}{dx}\,dx$ by du (and that simplifies things even further!). The resultant integral will have the form $\int H(u)\,du$ (where "H" is the hairy function) and be easier to integrate ... usually. Just remember:

$\boxed{\text{After substituting } u = g(x) \text{ and } du = g'(x)\,dx, \text{ replace every "x" by the appropriate function of "u"}}$

- $\int \frac{x^2+1}{x+1}\,dx = ?$ Let u = x + 1 so $du = \frac{du}{dx}\,dx = (1)\,dx = du$ and get $\int \frac{x^2+1}{u}\,du$ but we haven't got rid of all the x's yet, so we note that x = u - 1 and get, finally, $\int \frac{(u-1)^2+1}{u}\,du = \int (u - 2 + \frac{2}{u})\,du = \frac{u^2}{2} - 2u + \ln|u| + C$

  $= \frac{1}{2}(x+1)^2 - 2(x+1) + \ln|x+1| + C.$
- $\int$ ..

S: Hold on a minute. I don't get this at all. You can do a million examples and I *still* wouldn't get it. How can you let u = something then let $du = \frac{du}{dx}\,dx$? Are you cancelling the dx's, is that what you're doing? That isn't legal ... is it? I mean ...

P: Okay, that's a fair question. But remember, you can <u>always</u> check the answer by differentiating. If I say that $\int \frac{x^2+1}{x+1}\,dx = \frac{1}{2}(x+1)^2 - 2(x+1) + \ln|x+1| + C$, then just differentiate the right-side and see if you get $\frac{x^2+1}{x+1}$. See?

S: Aren't I supposed to understand what I'm doing or am I just supposed to turn some crank and out pops ...

P: Look here. I said that this arises from the Chain Rule: $\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$ which is equivalent to

$\int f'(u)\frac{du}{dx}\,dx = f(u) + C$. See the $\frac{du}{dx}\,dx$ in the integrand? Good. Now watch this. We'll change it into just du. It's clear that

$\int f'(u)\,du = f(u) + C$, right? (After all, if you differentiate f(u), then integrate it, you get back to f(u) ... except for a constant C.) Okay, this means that $\int f'(u)\frac{du}{dx}\,dx = \int f'(u)\,du$ and now see what's happened to the $\frac{du}{dx}\,dx$? It's now just du, so we bypass this rigmarole and simply replace du by $\frac{du}{dx}\,dx$ ... which is precisely what we've been doing.

S: And if I don't like it, I can just differentiate the right-side to prove it, right?

P: Sure. Remember, differentiation is a SCIENCE but integration is an ART ... perhaps a black art. Besides, checking your answer is what we like to do anyway, right? Besides, you can't argue with success ... and we've been pretty successful in evaluating the above integrals. Did I ever tell you the story of Oliver Heaviside?

S: Oh no, please don't.

P: This is interesting ... and won't be on the final exam, I promise. Heaviside was an electrical engineer who lived about the turn of the century (the 20th century). He invented a scheme for solving the darndest problems; it's called the Heaviside calculus. Mathematicians scoffed: "You can't do this!" and "How do you justify that?" and so on. His response? "Check my answer. It works."

S: I'm afraid to ask ... but give me an example.

P: Well ... let's see. Suppose we use the notation Dy in place of $\frac{dy}{dx}$, and D^2y in place of $\frac{d^2y}{dx^2}$, and so on. Then we see that $De^{ax} = a\,e^{ax}$ and $D^2e^{ax} = a^2e^{ax}$ and, continuing, $D^{100}e^{ax} = a^{100}e^{ax}$ so we can just *replace D by "a"* and get the umpteenth derivative of e^{ax}: $\boxed{D^n e^{ax} = a^n e^{ax}}$. "Just replace D by a". That's a Heaviside Rule. Also, if we use the notation $(D+2)\, e^{ax}$ we'd mean $D\, e^{ax} + 2\, e^{ax} = (a + 2)\, e^{ax}$ and $(D+2)^2 e^{ax}$ would mean $(D+2)(D+2)e^{ax} = (D+2)((a + 2)\, e^{ax}) =$ (performing the differentiation) $= (a^2 + 2a + 4)\, e^{ax}$ which is precisely $(a+2)^2 e^{ax}$. In other words, $(D+2)^2 e^{ax} = (a+2)^2 e^{ax}$ and again we just *replace D by "a"*. You can check it out for yourself. $(D+2)^3 e^{ax} = (a^3 + 6a^2 + 24a + 8)e^{ax} = (a+2)^3 e^{ax}$ and $\boxed{(D+2)^n e^{ax} = (a+2)^n e^{ax}}$ so on.

S: That's Heaviside calculus?

P: Part of it. Pay attention. Now we do something different. Suppose we differentiate a product. That is, we perform $D(e^{ax} u(x))$ where u(x) is some function of x. According to the PRODUCT rule we'd get $D(e^{ax} u) = e^{ax} u' + a\, e^{ax} u$ or, using our slick D-notation, we'd get: $D(e^{ax} u) = e^{ax} (u' + a\, u)$ which we'd write as $D(e^{ax} u) = e^{ax} (D+a)\, u$. Similarly $D^2(e^{ax} u) = e^{ax} (u'' + 2a\, u' + a^2 u) = e^{ax} (D+a)^2 u$. See the rule now? No? One more time then. We differentiate e^{ax} u three times and we'd get a factor e^{ax} in every term which we'd factor out and get: $e^{ax}(u''' + 3a\, u'' + 3a^2\, u' + a^2\, u)$ which is precisely $e^{ax}(D^3 u + 3aD^2 u + 3a^2 Du + a^2 u)$ which is precisely $e^{ax}(D+a)^3 u$. Now you see the rule:

$\boxed{D^n(e^{ax}u(x)) = e^{ax}(D+a)^n\, u(x)}$ which is called the Heaviside Shift Theorem. The e^{ax} gets shifted to the left, past the D, and *"a" gets added to D* leaving the operation $(D+a)^n$ to be performed on the function u(x).

S: That's Heaviside calculus?

P: Listen! There's more! Now if Dy means $\frac{dy}{dx}$ then $\frac{1}{D}$ y should mean ... what? Well, whatever $\frac{1}{D}$ y means, when we differentiate it via $D\left(\frac{1}{D} y\right)$ we should get y, right? After all, we just "cancel the D's". That means that $\frac{1}{D}$ y must be nobody else but $\int y\, dx$. Now, if we perform the operation $\frac{1}{D}\, e^{ax}$ we get $\int e^{ax}\, dx = \frac{e^{ax}}{a}$ (and we'll ignore the +C) then if we perform that operation again we'd get $\frac{1}{D}\frac{1}{D}\, e^{ax} = \frac{1}{D^2}\, e^{ax} = \frac{e^{ax}}{a^2}$ (again ignoring the constant of integration) . See? Just *replace D by "a"*. Now comes the good part.

S: Can I go now?

P: The best is yet to come! We want to evaluate $\int e^x x^2\, dx$ which Heaviside would write as $\frac{1}{D}\, e^x\, x^2$ and then he'd use his Shift Theorem to get $e^x \frac{1}{D+1}\, x^2$ (where we "shifted the e^x" and "added 1 to D"). What to do with $\frac{1}{D+1}\, x^2$? Well, we can divide D + 1 into 1 by long division (either that, or recognize $\frac{1}{1+D}$ as the sum of a geometric series with first term "1" and common ratio -D). In any case we'd get: $\int e^x x^2\, dx = \frac{1}{D}\, e^x\, x^2 = e^x \frac{1}{D+1}\, x^2 = e^x \left(1 - D + D^2 - D^3 + ...\right) x^2$ $= e^x \left(x^2 - 2x + 2 - 0 + 0 \,...\right)$ where we performed the indicated differentiations on x^2 and we'd conclude that $\int e^x x^2\, dx = e^x \left(x^2 - 2x + a^2\right) + C$ and we finally added the "C".

S: You gotta be kidding. Can you really do all that?

P: As Oliver would say: "Check my answer. It works."

S: Can we get back to "let u = *this* and du = *that*"?

P: First, I have one more technique to show you ... for evaluating indefinite integrals. This is perhaps the most important technique available and it's really very nice.

INTEGRATION BY PARTS:

We consider one last "Differentiation Rule", hoping it will result in an "Integration Rule" ... and the rule we consider is a familiar one: $\frac{d}{dx}\, uv = u\frac{dv}{dx} + v\frac{du}{dx}$ (the Product Rule for differentiation) . Integrating both sides with respect to x, we get an "Integration Rule": $\int\left(\frac{d}{dx}\, uv\right) dx = \int\left(u\frac{dv}{dx}\right) dx + \int\left(v\frac{du}{dx}\right) dx$. Now stand back. By

$\int\left(\frac{d}{dx}uv\right)dx$ we mean a function which, when differentiated, yields $\frac{d}{dx}$ uv and that's clearly uv itself. Our "Integration Rule" then becomes: $uv = \int\left(u\frac{dv}{dx}\right)dx + \int\left(v\frac{du}{dx}\right)dx$ which we can rearrange to read:

$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$. Before we say anything else, let's substitute some functions for u and v and see if this "Rule" is getting us anywhere.

Examples:

- Let u = x and v = sin x. Then we substitute $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \cos x$ and $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ becomes:

  $\boxed{\int x\cos x\,dx = x\sin x - \int \sin x\,(1)\,dx}$ which is very nice because $\int \sin x\,dx$ is easy, so we get

  $\int x\cos x\,dx = x\sin x + \cos x + C$, and we can verify that we do indeed have the antiderivative of x sin x because $\frac{d}{dx}(x\sin x + \cos x + C) = x\sin x$.

- Let $u = x^2$ and v = sin x. Then we substitute $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos x$ and $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ becomes:

  $\int x^2\cos x\,dx = x^2\sin x - \int \sin x\,(2x)\,dx$ or $\boxed{\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx}$. In other words, we've replaced the problem of evaluating $\int x^2\cos x\,dx$ by the problem of evaluating the simpler integral $\int x\sin x\,dx$ but, of course, we've already evaluated this one by the same technique so we can write, finally:

  $\int x^2\cos x\,dx = x^2\sin x - 2(x\sin x + \cos x) + C$ (adding the C after the last integral sign has gone).

- Let u = arctan x and v = x, so $\frac{du}{dx} = \frac{1}{1+x^2}$ and $\frac{dv}{dx} = 1$ so the "Rule" $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ becomes:

  $\boxed{\int \arctan x\,dx = x\arctan x - \int\frac{x}{1+x^2}dx}$ $= x\arctan x - \frac{1}{2}\,ln\,(1+x^2) + C$ (where we put $\int\frac{x}{1+x^2}dx = \frac{1}{2}\,ln\,(1+x^2)$ because we've already done this one, earlier, using our method of substitution).

S: Please, no more! Are you really thinking about starting with this ... this "Rule", $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$, then just plugging in functions for u and v and seeing what we get? I mean, is that what I'm supposed to ...

P: I just want to check to see if this Integration Rule which derives from the Product Rule for differentiation is going to be useful ... and we can see that it <u>is</u>, because how else would you integrate arctan x? In general, you'll run across $\int \arctan x\,dx$ and you'll have to figure out what is u and what is v and ...

S: You're kidding, right? I mean, you already knew the answer ... who's u and who's v ... <u>then</u> you invented the integral, yet you expect me to start with the integral and ...

P: Okay, that's a fair comment, so let's look at our new Integration Rule ... carefully:

INTEGRATION BY PARTS FORMULA

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

Comments:

- The original integral, having the form $\int u \frac{dv}{dx}\, dx$, is replaced by uv - $\int v \frac{du}{dx}$ which involves <u>another</u> integral ... hopefully simpler than the original! But we have done a <u>partial</u> integration, because "uv" <u>has</u> come out from under an $\int$ sign ... so this technique is called INTEGRATION by PARTS.
- The original integral involves the integration of a certain *product* : $(u)\left(\frac{dv}{dx}\right)$ ... which, in our examples above, were x cos x or x^2 cos x or (1) arctan x. After applying the Integration by Parts formula given above, we are <u>still</u> left with an integral, but now the two factors (u) and $\left(\frac{dv}{dx}\right)$ reappear transformed: the first factor, (u), reappears differentiated, $\left(\frac{du}{dx}\right)$, whereas the second factor, $\left(\frac{dv}{dx}\right)$, reappears integrated, (v), giving the remaining integral $\int (v)\left(\frac{du}{dx}\right) dx$. This transformation gives us the clue as to who's u and who's v. In fact, we want to choose them so the remaining integral is simpler.
- Since "u" reappears differentiated, we let u = *the factor we want to reappear differentiated*. What's left in the original integrand must then be $\frac{dv}{dx}$.

Examples: (a) $\int x \ln x\, dx = $ **?** (b) $\int x \arctan x\, dx = $ **?**

Solutions:

(a) Since *ln* x is a frightening (transcendental) function with a very nice derivative, we let u = *ln* x (because we'd rather have the derivative $\frac{du}{dx} = \frac{1}{x}$). That means that $\frac{dv}{dx} = x$ so v =
$\frac{x^2}{2}$ (and this will show up in the remaining integral, along with $\frac{du}{dx}$) . Having identified all <u>four</u> players in this drama: u, $\frac{du}{dx}$, v and $\frac{dv}{dx}$, we can substitute into the Integration by Parts formula $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}$ to get: $\int \ln x\; x\, dx$

$= (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\frac{1}{x}\, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2}\int x\, dx$ and you must admit that the remaining integral <u>is</u> pretty easy. Finally, then,

$\int x \ln x\, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$ (which we can verify by differentiating the right-side, to get x *ln* x).

(b) In $\int x \arctan x\, dx$, the function arctan x has a "nice" derivative so we let u = arctan x, hence $\frac{dv}{dx}$ must be x (since the original integrand must be $u\frac{dv}{dx}$ so if we know u we also know $\frac{dv}{dx}$). We need all <u>four</u> of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$ so we calculate $\frac{du}{dx} = \frac{1}{1+x^2}$ (the "nice" derivative) and $v = \frac{x^2}{2}$. Substituting into the "Formula" we get $\int x \arctan x\, dx$

$= \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$. We pause long enough to evaluate the remaining integral: $\int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1-1}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan x$ so we finally get:

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2}(x - \arctan x) + C.$$

S: Whoa! What happened there? I mean, $\frac{x^2}{1+x^2} = \frac{x^2+1-1}{1+x^2}$ and all that jazz?

P: Cute, eh? I just added and subtracted "1" from the numerator, then split it up into two pieces, each being easy to integrate, namely $\frac{x^2+1}{1+x^2}$ (which, in case you didn't know, is "1") and $\frac{1}{1+x^2}$ whose integral I've memorized. Do you like that?

S: No! Besides, you said you let u = arctan x because it has a "nice" derivative. There's another factor in $\int x \arctan x \, dx$ and it's x and it has an even "nicer "derivative so I'd let u = x.

P: And what does that leave for $\frac{dv}{dx}$?

S: Huh?

P: The original integrand is x arctan x which is $u \frac{dv}{dx}$. If u = x, then $\frac{dv}{dx}$ must be arctan x. Hence, to find v itself, you'd have to integrate arctan x. See? My way, letting u = arctan x, eliminates the need for integrating arctan x.

S: But we know $\int \arctan x \, dx$ from an earlier example ... it's ... uh, x arctan x - $\frac{1}{2}$ *ln* $(1+x^2)$, so doing it my way, I'd let u = x and $\frac{dv}{dx}$ = arctan x so $\frac{du}{dx}$ = 1 and v = $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2)$ and I'd use the magic integration-by-parts formula and get: $\int x \arctan x \, dx = x\left(x \arctan x - \frac{1}{2} \ln(1+x^2)\right) - \int \left(x \arctan x - \frac{1}{2} \ln(1+x^2)\right) dx$ ... mamma mia, that's not so nice ... uh, I leave it as an exercise for the prof.

It is convenient, for purposes of "memorizing the formula", to rewrite $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ in the form $\boxed{\int \mathbf{u\, dv = u\, v} - \int \mathbf{v\, du}}$ where we've replaced $\frac{dv}{dx}$ dx by dv and $\frac{du}{dx}$ dx by du. (We've done this *replacement* before, in our "method of substitution".) Although easier to remember, I don't think it's easier to use!

Example: $\int x^2 \ln x \, dx = ?$

Solution: Let u = *ln* x, then $\frac{du}{dx} = \frac{1}{x}$. What's left under the integral sign is x^2 dx which must be dv, that is, dv = x^2 dx so v = $\int x^2 dx = \frac{x^3}{3}$. Now we use $\int u \, dv = uv - \int v \, du$ and get $\int x^2 \ln x \, dx = \ln x \left(\frac{x^3}{3}\right) - \int \left(\frac{x^3}{3}\right) \frac{1}{x} dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$ (which, as usual, can be verified by differentiating the right-side to get x^2 *ln* x).

S: I like $\int u \, dv = uv - \int v \, du$ much better than $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$.

P: Then use it! In fact, when I was your age my calculus prof gave us the formula as:

$\boxed{\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int f'(x) \int g(x)\, dx\, dx}$ which scared me, I admit, but now that I'm much older and a little wiser I see what it's saying: if the original integral has a product, f(x) g(x), you use Integration by Parts and get another integral where the first factor reappears differentiated, that's f'(x), and the second reappears integrated, that's $\int g(x)\, dx$. If you remember that, then you can determine "who's u and who's v" in your head. For example, suppose I wanted to integrate $\int x \arcsin x\, dx$. I could use integration by parts and get two possible "second integrals", either

$\int \left(\frac{x^2}{2}\right)\left(\frac{1}{\sqrt{1-x^2}}\right) dx$ where the 2nd factor was differentiated, or $\int (1)\left(\int \arcsin x\, dx\right) dx$ where the 1st factor was differentiated. I'd choose the first type, letting u = *the factor I want to reappear differentiated*, namely u = arcsin x.

Similarly, for $\int x\, e^x\, dx$ I could use Integration by Parts and replace this integral with $\int \left(\frac{x^2}{2}\right)(e^x)\, dx$ ... the integrand is (*the integral of* x) (*the derivative of* e^x) ... OR with $\int (1)(e^x)\, dx$ where the integrand is (*the derivative of* x) (*the integral of* e^x))). I'd choose the latter, letting u = *the factor I want to reappear differentiated*, namely u = x.

Assorted Examples in Integration:

- For $\int \frac{x-3}{x^2+2x+1}\, dx = \int \frac{x-3}{(x+1)^2}\, dx$, let u = x+1, du = $\frac{du}{dx}$ dx = dx and get $\int \frac{u-4}{u^2}\, du = \int \left(\frac{1}{u} - \frac{4}{u^2}\right) du$ $= ln|u| + \frac{4}{u} + C = ln|x+1| + \frac{4}{x+1} + C$
- For $\int \frac{x^{1/3}}{1+x^{2/3}}\, dx$, let u = 1 + $x^{2/3}$ so du = $\frac{2}{3}\, x^{-1/3}$ dx so dx = $\frac{3}{2}\, x^{1/3}$ du and get $\int \frac{x^{1/3}}{u} \frac{3}{2} x^{1/3}\, du = \frac{3}{2} \int \frac{x^{2/3}}{u}\, du = \frac{3}{2} \int \frac{u-1}{u}\, du = \frac{3}{2}(u - ln\,|u|) + C = \frac{3}{2}\left(1 + x^{2/3} - ln\,(1 + x^{2/3})\right) + C$

S: Wait! I thought we look for a function whose derivative is "next to dx" and let u = this function and so on. But I don't see that *that's* what you're doing.

P: Okay, let me give you another technique: look for the hairiest function around and u = that. Then, by hook or crook, get rid of every "x" and "dx" and hope the resultant integration in terms of "u" and "du" is easier. In the first example above, after recognizing $x^2+2x+1 = (x+1)^2$, I didn't like $(x+1)^2$ in the denominator ... it'd be much easier if it were simply x^2 living there ... so I let u = x+1 and get a u^2 in the denominator. And replacing the numerator and the "dx" by the appropriate expressions in "u" didn't make things any worse ... and I got an easy integral in terms of "u". See? In the next example I didn't like the $x^{2/3}$ so I *could* have let u = $x^{2/3}$ and du = $(2/3)x^{-1/3}$dx and get $\int \frac{x^{1/3}}{1+u} \frac{3}{2} x^{1/3} du$ but then I'd get 1+u in the denominator which I'd like to replace by a single variable so I'd continue with a second substitution, letting v = 1+u and so on ... hence I thought it'd be easier to simply let u = 1 + $x^{2/3}$ right away and ..

S: That's confusing. How would *I* know to do that?

P: Experience. Do dozens of integrals and you eventually get the hang of it. Remember, integration is an art. You have to have a feel for it and you get this "feel" by doing lots of examples ... yourself ... not just watching me or reading the examples in some book. Remember, too, that math is not a spectator sport. Get involved. Anyway, let me say something about the last example above. I eventually eliminated all references to "x" and "dx" and got an easy integral in "u" and "du": $\frac{3}{2} \int \frac{u-1}{u}\, du = \frac{3}{2}(u - ln\,|u|) + C = \frac{3}{2}\left(1 + x^{2/3} - ln\,(1 + x^{2/3})\right) + C$ which I would most likely write as $\frac{3}{2}\left(x^{2/3} - ln\,(1 + x^{2/3})\right) + C$ absorbing the constant $\frac{3}{2}(1)$ into the arbitrary constant C.

S: And you forgot the absolute value signs in the *ln* !

P: Well, actually, I didn't because $| 1 + x^{2/3} | = 1 + x^{2/3}$ since this function is always positive.
S: How would I know that?
P: $x^{2/3}$ is a square. It's the square of $x^{1/3}$. That makes it positive, or at least non-negative. *Everybody* knows that ... even you.
S: Can you do an example which doesn't work out? I mean, sometimes you make a mistake, right? Sometimes, you let u = *this* and you should have let u = *that*.

Example: Evaluate $\int \frac{\ln x}{e^x + 1}\, dx$ ... if you can.

- Try $u = e^x + 1$ so $e^x = u - 1$ hence $x = \ln (u - 1)$ (taking the *ln* of each side) and $dx = \frac{dx}{du}\, du = \frac{1}{u-1}\, du$ then we'd get $\int \frac{\ln \ln (u - 1)}{u} \frac{1}{u - 1}\, du$ which actually seems worse that the original integral.
- Now let's try $u = \ln x$, then $x = e^u$ and $dx = \frac{dx}{du}\, du = e^u\, du$ and we'd get $\int \frac{u}{e^{e^u}+1} e^u\, du$ which is awful!
- Let's try ... let's try ... uh ...

S: Give up?
P: Yeah, I can't do this one.
S: Welcome to the club.
P: That's not surprising, you know. If you just invent some function, f(x), chances are I couldn't evaluate $\int f(x)\, dx$ in terms of known functions. But I could always invent a brand new function. In fact, let me invent the following function: $P(x) = \int_1^x \frac{\ln t}{e^t+1}\, dt$ then what do you think $\frac{d}{dx} P(x)$ is?
S: I haven't the foggiest.
P: Remember the Fundamental theorem, and how we managed to evaluate definite integrals without resorting to Riemann SUMS? In trying to evaluate $\int_a^b f(x)\, dx$ we *invented* the function $A(t) = \int_a^t f(x)\, dx$ and discovered that $\frac{d}{dt} A(t) = f(t)$. We should make a fuss about this so you'll remember it:

$$\boxed{\frac{d}{du} \int_a^u f(z)\, dz = f(u)}$$

In fact, I even changed the labels on the variables so you'd recognize it even if the variables aren't called "x" and "t". For example:

- If $H(w) = \int_0^w e^{\sin t}\, dt$ then $H'(w) = e^{\sin w}$.
- If $G(k) = \int_{-1}^k \tan x^2\, dx$ then $\frac{d}{dk} G(k) = \tan k^2$.
- $\frac{d}{dx} \int_\pi^x \ln \ln t\, dt = \ln \ln x$.
- If $P(x) = \int_1^x \frac{\ln t}{e^t+1}\, dt$ then $\frac{d}{dx} P(x) = \frac{\ln x}{e^x + 1}$. Hence, I now have a function whose derivative is $\frac{\ln x}{e^x + 1}$ just as I

promised. Hence, $\int \frac{ln\ x}{e^x + 1}\,dx = P(x) + C$.

S: You can't do that! Just because you gave it a name ... called it P(x) ... does that mean you've actually evaluated this integral? Can *I* do that too? I mean, if you give me some integral I can't evaluate, can I just call it P(x) ... and will I get full marks on an exam?

P: No. I'll only give you integrals which can be evaluated in terms of known functions, and that's how I want you to evaluate them. I'm only telling you all this ... inventing new functions ... so you'll see that it can be done and you won't ever say "this integral can't be evaluated" when you really mean "this integral can't be evaluated in terms of known functions". Let me see you evaluate $\int \frac{dx}{x}$.

S: Easy. It's $ln\,|\,x\,| + C$.

P: And before the natural log function was invented, what would you say about this integral?

S: Uh ... it can't be evaluated in terms of known functions, right?

P: Right. Now suppose I define $P(x) = \int_1^x \frac{ln\ t}{e^t+1}\,dt$ ... which is, after all, the area associated with the curve $y = \frac{ln\ t}{e^t+1}$ from t = 1 to t = x ... and if I determined this area very accurately for each x (using, for example, Riemann sums with plenty of rectangles) and if I made a table of values of P(x) and gave this function a name, say the Ponzo function, and I found where it was increasing and decreasing (because, you see, I know the derivative of P(x)), and I managed to get it into a bunch of calculus books ... then, in future, when students ran across the above integral they'd just write the answer as P(x) + C because P(x) would be a "known function". See? It's just like writing answers $ln\,|\,x\,| + C$ or $\sin x + C$ or $e^x + C$. See? You're happy only because somebody has already invented the *ln* and sine functions.

S: I guess so.

Examples:

(a) $\frac{d}{dt}\left(\int_t^{t^2} ln\,(1 + x^2)\,dx\right) = \mathbf{?}$

(b) $\int ln\ t\,dt = \mathbf{?}$

(c) $\int_e^{e^2} \frac{ln(ln\ x)}{x}\,dx = \mathbf{?}$

Solutions:

(a) $\int_t^{t^2} ln\,(1+x^2)dx = \int_0^{t^2} ln\,(1+x^2)dx - \int_0^{t} ln\,(1+x^2)dx$ (using a property of definite integrals)

$= \int_0^{u} ln\,(1+x^2)dx - \int_0^{t} ln\,(1+x^2)dx$ where $u = t^2$. Then $\frac{d}{dt}$ gives

$\frac{d}{du}\left(\int_0^{u} ln\,(1+x^2)dx\right)\frac{du}{dt} - \frac{d}{dt}\left(\int_0^{t} ln\,(1+x^2)dx\right)$ (where we've used the Chain Rule)

$= ln(1+u^2)\,(2t) - ln(1+t^2) = 2t\,ln(1+t^4) - ln(1+t^2)$

Note: In the above, we want to use $\frac{d}{du}\int_a^u f(z)\,dz = f(u)$ but this requires:

(i) the lower limit must be constant ... and that's the reason for writing $\int_t^{t^2} = \int_0^{t^2} - \int_0^t$

(ii) the upper limit is a variable,

(iii) we're differentiating with respect to this variable upper limit (that's why we put $u = t^2$)

(b) In $\int \ln t\, dt$, we'll use $\int u\, dv = uv - \int v\, du$ and we'll let $u = \ln t$ so $dv = dt$. Then $du = \frac{du}{dt}\, dt = \frac{1}{t}\, dt$ and $v = \int dt = t$ hence we get $uv - \int v\, du = t \ln t - t + C$ (which we verify by differentiating).

(c) $\displaystyle\int_e^{e^2} \frac{\ln(\ln x)}{x}\, dx = \Big[\ln x \ln(\ln x) - \ln x\Big]_e^{e^2}$ (we've already evaluated the indefinite integral, earlier)

$=(\ln e^2 \ln(\ln e^2) - \ln e^2) - (\ln e \ln(\ln e) - \ln e) = (2 \ln 2 - 2) - (0 - 1) = 2 \ln 2 - 1$

When we evaluate a <u>definite</u> integral (as opposed to an <u>indefinite</u> integral), we must first find an antiderivative of the integrand (i.e. the indefinite integral), then evaluate at it the upper limit, then at the lower limit, then subtract. If we make a substitution, $u = g(x)$ say, then integrate, we get a function of u so we have to go back to x again to evaluate at the upper and lower limits. This is sometimes inconvenient, especially if you make more than one substitution to determine the antiderivative. It's nice to know, then, that you can carry your limits along with you. Just ...

S: I haven't the faintest idea what you're talking about. Can't you give me an example?

Consider $\displaystyle\int_0^1 \frac{x}{1+x^2}\, dx$. We make the substitution $u = 1+x^2$ so that $du = \frac{du}{dx}\, dx = 2x\, dx$, hence $x\, dx = \frac{1}{2}\, du$ and we get $\displaystyle\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2} \ln |u|$ (and we omit the constant of integration because we want to evaluate a <u>definite</u> integral). Now we return to the original variable, x, and get: $\displaystyle\int_0^1 \frac{x}{1+x^2}\, dx = \frac{1}{2} \ln (1+x^2)]_0^1 = \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1$.

HOWEVER (and here's the nice part) we could also have stuck with the "u" variable, changed the limits and written $\frac{1}{2} \ln |u|\,]_1^2 = \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1$. See how it works? When $x = 0$, then $u = 1+0^2 = 1$ and when $x = 1$ then $u = 1+1^2 = 2$ so the u-limits are 1 and 2. In other words, when we changed variables (from x to u) we can also *change limits*, writing:

$$\boxed{\int_0^1 \frac{x}{1+x^2}\, dx = \frac{1}{2}\int_1^2 \frac{du}{u}}$$ etc. etc.

Examples: Evaluate: (a) $\displaystyle\int \frac{dx}{1-6x+9x^2}$ (b) $\displaystyle\int_1^2 \frac{dx}{1-6x+9x^2}$

(c) $\displaystyle\int_{-\pi}^{\pi} \sin t^3\, dt$ (d) $\displaystyle\int \frac{x^2}{1+x^2}\, dx$ (e) $\displaystyle\int_0^{\pi/4} \sec^2 t \sqrt{1-\tan^2 t}\, dt$

(f) $\displaystyle\int_1^e \frac{\ln^2 t}{t}\,dt$ (g) $\displaystyle\int_{1/2}^1 \frac{dx}{\sqrt{2x-x^2}}$ (h) $\displaystyle\int \frac{\tan\sqrt{x}}{\sqrt{x}}\,dx$

(i) $\displaystyle\frac{d}{dz}\left(\int_0^{\arcsin z} e^{\sin t}\,dt\right)$

Solutions:

(a) $\displaystyle\int \frac{dx}{1-6x+9x^2} = \int \frac{dx}{(1-3x)^2}$. Let u = 1-3x so du = -3 dx and get $\displaystyle\frac{-1}{3}\int \frac{du}{u^2} = \frac{-1}{3}\left(\frac{-1}{u}\right) + C = \frac{1}{3(1-3x)} + C$

(b) $\displaystyle\int_1^2 \frac{dx}{1-6x+9x^2} = \int_1^2 \frac{dx}{(1-3x)^2}$. Let u = 1-3x so du = -3 dx. Also, when x goes from 1 to 2, u goes from -2 to -5 so we get $\displaystyle\int_1^2 \frac{dx}{(1-3x)^2} = \frac{-1}{3}\int_{-2}^{-5} \frac{du}{u^2} = \frac{-1}{3}\left[\frac{-1}{u}\right]_{-2}^{-5} = \frac{-1}{3}\left\{\left(\frac{1}{5}\right) - \left(\frac{1}{2}\right)\right\} = \frac{1}{10}$.

(c) $\displaystyle\int_{-\pi}^{\pi} \sin t^3\,dt = 0$ because $\sin t^3$ is an ODD function.

(d) $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int\left(1 - \frac{1}{1+x^2}\right)dx = x - \arctan x + C$

S: You've already done that one.
P: I wanted you to see "add and subtract" again. Pay attention.

(e) $\displaystyle\int_0^{\pi/4} \sec^2 t\sqrt{1-\tan^2 t}\,dt$. Let x = tan t so dx = $\frac{dx}{dt}$ dx = $\sec^2 t$ dt (we make this substitution because we see not only tan t but also its derivative, $\sec^2 t$). When t goes from 0 to $\pi/4$, x goes from tan 0 = 0 to tan $\pi/4$ = 1 so we get $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ which we recognize as the area under a quarter-circle of radius 1 (since $y = \sqrt{1-x^2}$ is the upper half of the circle $x^2+y^2=1$, and for x between 0 and 1 we get the area of a 1/4-circle) namely $\frac{\pi}{4}\ \pi 1^2 = \frac{\pi}{4}$.

(f) $\displaystyle\int_1^e \frac{ln^2 t}{t}\,dt$. Let u = *ln* t so du = $\frac{dt}{t}$ and when t goes from 1 to e, u goes from *ln* 1 = 0 to *ln* e = 1. We get

$$\int_1^e \frac{ln^2 t}{t}\,dt = \int_0^1 u^2\,du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}\ .$$

(g) $\displaystyle\int_{1/2}^1 \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$. Let u = x-1 and du = dx. For x = $\frac{1}{2}$, u = - $\frac{1}{2}$ and for x = 1, u = 0 so we get

$$\int_{1/2}^{1} \frac{dx}{\sqrt{2x-x^2}} = \int_{-1/2}^{0} \frac{du}{\sqrt{1-u^2}} = \left[\arcsin u\right]_{1/2}^{0} = \left(\arcsin 0\right) - \left(\arcsin(-1/2)\right) = 0 - (-\frac{\pi}{6}) = \frac{\pi}{6}$$

S: That's pretty tricky. Should I know all this? I'll never remember it.
P: Completing the squares is useful. Remember it! See how many examples I'm doing? You'll get it by osmosis.

(h) $\int \frac{\tan\sqrt{x}}{\sqrt{x}} dx$. Let $u = \sqrt{x}$ so $du = \frac{du}{dx} dx = \frac{dx}{2\sqrt{x}}$ and we get $2 \int \tan u \, du = 2 \, ln \, |\sec u| + C$

(i) $\frac{d}{dz}\left(\int_0^{\arcsin z} e^{\sin t} dt\right)$. Let u = arcsin z so (using the Chain rule) we get $\frac{d}{dz}\left(\int_0^{u} e^{\sin t} dt\right) = \frac{d}{du}\left(\int_0^{u} e^{\sin t} dt\right)\frac{du}{dz} =$

$e^{\sin u} \frac{1}{\sqrt{1-u^2}} = e^{u} \frac{1}{\sqrt{1-\arcsin^2 z}}$.

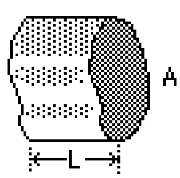
LECTURE 18

APPLICATIONS OF THE DEFINITE INTEGRAL

VOLUMES:

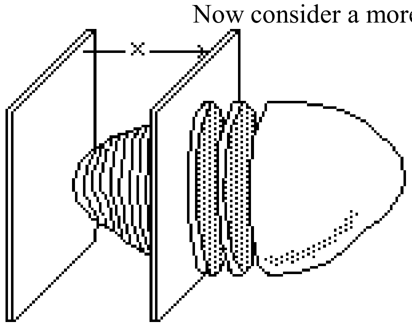
Now that we're experts in integration, we should evaluate some integrals which represent something other than areas.

Consider the problem of determining the volume of a solid. If it were a right-circular cylinder we'd have a formula: $\pi r^2 h$ (where r = the radius, h = the length). If it were a sphere we'd also have a formula: $4\pi r^3/3$. In fact, there are many solids for which we have formulas ... BUT, we need a method in cases where the solid is irregular. Just as we did when we considered the problem of the area enclosed by curves, we subdivide the volume into simpler volumes for which we <u>do</u> have a formula. (Remember the rectangles, both vertical and horizontal ... and the circular sectors for areas enclosed by polar curves?)



Before we go on, let's consider the volume generated by a plane area, A, which moves a distance L in a direction perpendicular to the plane. The volume generated is AL. If the plane area is a circle, then $A = \pi r^2$ and the volume of the "right-circular cylinder" is, as we've seen, $\pi r^2 h$. That's just a special case. For <u>any</u> plane area A, the solid is *still* called a cylinder and the volume is AL.

Now consider a more general solid.



Perhaps the simplest method is to cut the solid into many very thin slices. Each slice is very nearly cylindrical with some cross-sectional area, say A, and a thickness, say h. The volume of such a slice is (very nearly) Ah. To make things more precise, we'll assume that the volume is cut by a plane (or a knife) at a distance x from some fixed plane. The cross-sectional area will, of course, depend upon x. We'll call it A(x) and the thickness of the slice, at the place x, we'll call dx (which is a better name, for now, than "h"). Then A(x) dx is a good approximation to the volume of the thin slice. If we SUM the volumes of all such slices we'll get the total volume of the solid, and the definite integral does the $\int$umming for us.

We get $\boxed{V = \int_a^b A(x)\,dx}$. In other words, the volume of a solid is the integral of the cross-sectional area.

S: You're going too fast. I don't see how you'll get the exact volume of the solid. Maybe you'll get an approximation, but the exact volume? Not likely!

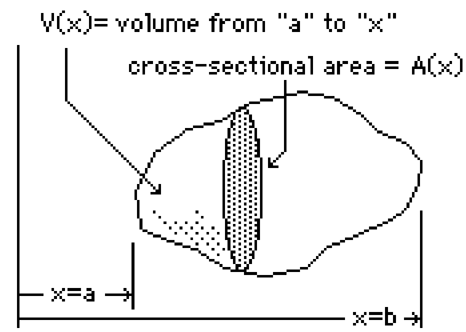
P: Okay, we'll do this from scratch. (Aren't you tired of constantly going back to Riemann SUMs?) The volume lies between x = a and x = b (where "x" measures the distance of the slices from some fixed plane) so now we subdivide the interval [a,b] into n subintervals of length $h = \frac{b-a}{n}$ and in each subinterval we pick a convenient value of x, say x_1 in the first subinterval, x_2 in the second and so on (does this sound familiar?) and now we consider the volume of the first slice of width h and cross-sectional area $A(x_1)$, the second slice of width h and cross-sectional area $A(x_2)$ and so on and the SUM of volumes of all such slices is $A(x_1)\,h + A(x_2)\,h + \ldots A(x_n)\,h = \sum_{k=1}^{n} A(x_k)\,h$ which we recognize as a Riemann SUM so now we let $n \to \infty$ so $h \to 0$ (does this sound familiar?) and get, in the limit, $\int_a^b A(x)\,dx$, the total volume of the solid. See?

S: Well ... not really.

P: Okay, we'll do it differently. Suppose we let A(x) be the cross-sectional area at the place x and let V(x) be the volume lying to the left of x. (i.e. from "a" to "x".) Now we'll differentiate V(x) via **Error!**. Note that V(x+h) - V(x) is just the volume from x to x+h and we can write:

$$A_{minimum}\,h < V(x+h) - V(x) < A_{maximum}\,h$$

where $A_{minimum}$ is the minimum and $A_{maximum}$ the maximum cross-sectional area between x and x+h. What we've done is found a cylindrical volume



bigger and smaller than V(x+h) - V(x). Now ...

S: Hey ... let me do it! This does seem familiar! We SQUEEZE, right? Okay ... uh, let's see ... we write

$$A_{minimum}\,h < V(x+h) - V(x) < A_{maximum}\,h$$

then $$A_{minimum} < \frac{V(x+h) - V(x)}{h} < A_{maximum}$$

then $$\lim_{h \to 0} A_{minimum} < \lim_{h \to 0} \frac{V(x+h) - V(x)}{h} < \lim_{h \to 0} A_{maximum}$$

then ... uh ...

P: Great! Keep going! The guy in the middle is just $\frac{dV}{dx}$ and now you've got to guarantee that the limits on each side are the same. That's how we SQUEEZE.

S: Are they the same? I mean, what is the limit of $A_{minimum}$?

P: Remember when we did this before, for areas? Go back and look at your notes. The method is exactly the same. The limits of $A_{minimum}$ and $A_{maximum}$ are just A(x) because they're the cross-sectional areas at some points between x and x+h and h is going to zero so ...

S: Yeah, I got it, but we'd get $A(x) < \lim_{h \to 0} \frac{V(x+h) - V(x)}{h} < A(x)$, or maybe $A(x) < \frac{dV}{dx} < A(x)$ and how does that give us $\frac{dV}{dx}$?

P: When taking limits we'd get $A(x) \le \frac{dV}{dx} \le A(x)$ and that makes $\frac{dV}{dx} = A(x)$.

S: How'd the < change all of a sudden to ≤ ?

P: Hmmm ... good question. Let's do this for something simpler. If x is a number less than 1, say $0 < x < 1$, then it's greater than 0 and smaller than x^2. (If you don't believe me, try a few squares and convince yourself that $x < x^2$.) Okay, we then have $0 < x < x^2$. Now let $x \to 0$. Do you think that $0 < \lim_{x \to 0} x < \lim_{x \to 0} x^2$. No. In fact all three are precisely 0, so when taking limits across inequalities we can't guarantee that < doesn't change to ≤, so we always write ≤ instead of <, just in case ... sort of insurance. In our case we'd get $A(x) \le \frac{dV}{dx} \le A(x)$ hence V(x) is an antiderivative of A(x). In fact, it's the special antiderivative that has the value 0 when x = a ... and that makes $V(x) = \int_a^x A(t)\,dt$. See?

$\frac{dV}{dx} = \frac{d}{dx} \int_a^x A(t)\,dt = A(x)$ (Remember? It's the integrand evaluated at the upper limit.) In particular, the total volume of our solid is V(b) or $\int_a^b A(t)\,dt$ which, of course, is the same as $\int_a^b A(x)\,dx$.

S: I meant to ask you about that. You seem to rather cavalier with what you call the variable under the integral sign. You call it "t", then you call it "x" and sometimes you even ...

P: $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$. Remember? Now watch this: $\int_a^b f(t)\,dt = \left[F(t)\right]_a^b = F(b) - F(a)$ and $\int_a^b f(z)\,dz = \left[F(z)\right]_a^b = F(b) - F(a)$. See? They're all the same number. It doesn't matter what you call the variable of integration. When the smoke clears, it's gone anyway. The variable of integration is sometimes called a "dummy variable". See?

S: Then why don't you always use the same name, like x ... which I like best.

P: Well ... uh, that's a good question. You see, it's like this. Nobody likes to write $\int_a^x f(x)\,dx$ because the x in the upper limit gets confused with the x under the integral. The upper limit is a particular value of the other x. You see ... uh, it's much better to distinguish between them and write $\int_a^t f(x)\,dx$ so we're considering the area from x = a to x = t (and it'd be bad manners to write "from x = a to x = x").

S: I've heard lots of people say that: "from x = a to x = x".

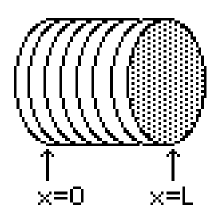
P: Well, you won't hear <u>me</u> say it!

S: But didn't you say ... when we were slicing that solid ... that we measure distance and call it x and slice the solid at the place x, and that'd mean at x = x, wouldn't it?

P: Uh ... I think we should go on. We've still got a ways to go before the end of this lecture. Pay attention.

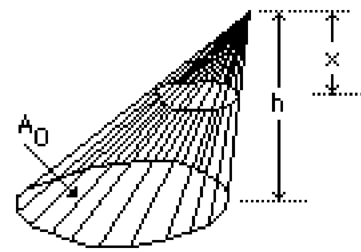
Examples:

Volume of a cylinder: Assume the cylinder lies between x = 0 and x = L and we slice it by planes perpendicular to the axis of the cylinder. The cross-sectional area is constant at some value A. Then $V = \int_0^L A\,dx = \left[Ax\right]_0^L = AL$, as expected. For a right circular cylinder, $A = \pi r^2$ and the volume is $V = \pi r^2 L$.



Volume of a cone:

The area A_0 lies in a plane. A point P lies a distance h above the plane. Every point on the perimeter of the area is joined to P, forming a cone. We slice the cone by planes parallel to the plane of the area, a distance x from the vertex of the cone. The cross-sectional area is a scaled-down version of the base area and has the value $A(x) = \left(\frac{x}{h}\right)^2 A_0$. The volume is $V = \int_0^h A(x)\,dx = \frac{A_0}{h^2}\int_0^h x^2 dx = \frac{A_0}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{1}{3} A_0 h$. If it's a right circular cone, $A_0 = \pi r^2$ and the volume is $V = \frac{1}{3}\pi r^2 h$, as expected.



S: Wait. The cross-section has area $A(x) = \left(\frac{x}{h}\right) A_0$, not $\left(\frac{x}{h}\right)^2 A_0$. How do you get ...

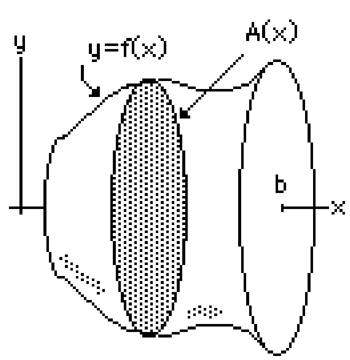
P: Suppose the area is a circle, so $A_0 = \pi r^2$. If we slice half-way from vertex to base the cross-section is again a circle with what radius?

S: I'd say $\frac{1}{2}$ r.

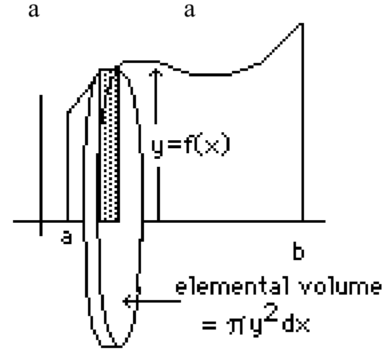
P: Right! And what's the area of a circle of radius $\frac{1}{2}$ r? It's $\pi\left(\frac{r}{2}\right)^2 = \frac{1}{4}\pi r^2$. See? It's $\frac{1}{4}$ of the base area, not $\frac{1}{2}$. In fact, if we scale every length by a factor $\frac{1}{3}$ then an area will be $\frac{1}{9}$ as large. If we scale down a solid so every length is reduced to $\frac{1}{3}$ then the <u>volume</u> is reduced to $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$. See how it works? If every metre of length is reduced by a certain factor then every quantity that's measured in metres^2 (like SURFACE AREA or CROSS-SECTIONAL AREA) gets reduced by that factor <u>squared</u>. Quantities measured in metres^3 (like VOLUME) get reduced by that factor <u>cubed</u>. Anyway, we see that the well-known formula for the volume of a cone, namely $\frac{1}{3}$(base area) (height) holds for *any* cone, not just right circular cones. Nice, eh?

S: If you say so.

VOLUMES OF SOLIDS OF REVOLUTION:



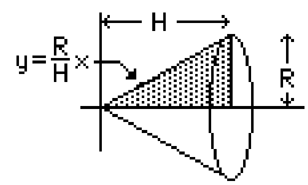
The simplest solids to deal with are clearly those which have particularly simple cross-sectional areas (since we have to integrate this to get the volume). One class of solids which has this nice feature is *Solids of Revolution*. We begin with an area in the x-y plane, enclosed by given curves ... for now we assume that the area is bounded by y = f(x), x = a, x = b and the x-axis (y = 0) as shown. Then we revolve this area about the x-axis, sweeping out a solid of revolution. Now, when we slice this volume by planes perpendicular to the x-axis the cross-sections are just circles! Indeed, if we locate the plane by its distance from the y-axis (x = 0), the radius of the circular cross-section is just y = f(x) so $A(x) = \pi y^2$ and if the thickness of each slice is dx then volume of the slice at the place "x" is just $A(x)\,dx = \pi\, y^2\, dx$. The total volume of the solid of revolution is then: $V = \int_a^b \pi y^2\, dx = \pi \int_a^b f^2(x)\, dx$



A nicer way to think of this is to avoid revolving the *entire* area at one time ... just subdivide the area into vertical rectangles and revolve these one at a time. Each such rectangle sweeps out a disc of radius f(x) (since it's located at the place x and the y-coordinate *is* the radius) and thickness dx. The volume is then $\pi y^2\, dx$ and $\int$umming all such disc-volumes (for every elemental rectangle, from x = a to x = b) gives $V = \pi \int_a^b f^2(x)\, dx$ as before. The advantage of this interpretation of the formula $\pi \int_a^b f^2(x)\, dx$ is that we can also consider subdividing the area into *other* elemental areas (such as horizontal rectangles!) and revolving each.

Example: Determine the volume of a cone of height H and base radius R.

Solution: We recognize the cone as a solid of revolution: the area enclosed by $y = \frac{R}{H}x$, x = 0, x = H, y = 0 is revolved about the x-axis. The volume is then:



$$V = \pi \int_0^H y^2 dx = \pi \left(\frac{R}{H}\right)^2 \int_0^H x^2 dx = \pi \left(\frac{R}{H}\right)^2 \left[\frac{x^3}{3}\right]_0^H = \frac{1}{3}\pi R^2 H$$

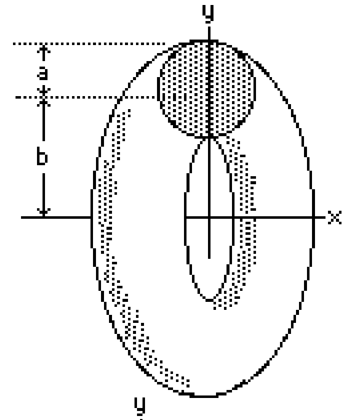
S: You just did a cone. Do you like cones?

P: Yes ... now pay attention.

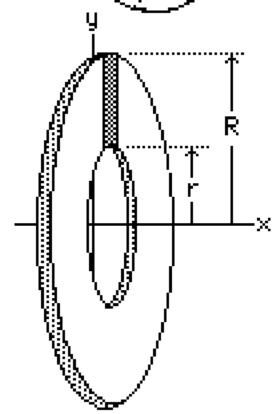
Example: Determine the volume of a sphere.

Solution: We first recognize that a sphere is a solid of revolution: just revolve the semi-circle $y = \sqrt{a^2-x^2}$ about the x-axis, from x = -a to x = a. The volume is $V = \pi \int_{-a}^{a} y^2 dx = \pi \int_{-a}^{a} (a^2-x^2)dx = \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3$

Example: Calculate the volume of a torus (or donut).



Solution: To generate a torus we revolve the area within a circle, $x^2+(y-b)^2 = a^2$, about the x-axis. We can solve this for $y = b \pm \sqrt{a^2-x^2}$, giving an upper half-circle and a lower half-circle. Now we subdivide the area into vertical rectangles (located at the place x, with thickness dx) and revolve each, generating a disc-with-a-hole (or a "washer"). If the outer radius is R and the inner radius r, the washer has volume $(\pi R^2-\pi r^2)$ dx (since the cross-sectional area is the area between two circles). But R is the y-value of the upper half-circle: $R = b + \sqrt{a^2-x^2}$ and r is the y-value of the lower half-circle: $r = b - \sqrt{a^2-x^2}$, hence the volume of the washer is:
$\pi(R^2 - r^2)\, dx = \pi(R - r)(R + r)dx = 4\pi b \sqrt{a^2-x^2}$ dx and the total volume of the torus is the $\int$um from x = -a to x = a, giving: $V = 4\pi b\int_{-a}^{a}\sqrt{a^2-x^2}dx$.



Although this integral seems difficult, we recognize it as the area beneath a semi-circle of radius "a", namely $\frac{1}{2}\pi a^2$. Finally, then, $V = 2\pi^2 ba^2$ which certainly has the dimensions of volume: if "a" and "b" are measured in metres, then ba^2 is in metres^3 as it should be. Also, if a = 0 we get V = 0 as expected (and that gives us more confidence in our answer). Finally (and this is really interesting), we can write this volume as (πa^2) $(2\pi b)$ which is just the area of the circle (πa^2) multiplied by the distance moved by its centre (around a circle of radius b, hence a distance $2\pi b$) ... and *that* gives us even more confidence in our answer.

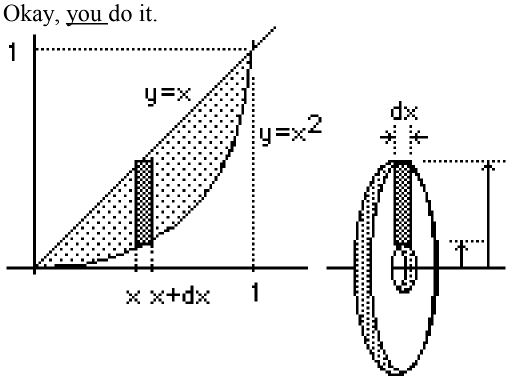
S: Why do you need all this *confidence*? Don't you have any faith in your answer?
P: I have faith only if the answer seems reasonable ... and this answer *does* seem reasonable.
S: I like that last thing you said: the volume is just the area multiplied by the distance ... uh, how did it go?
P: Revolve an area about a line (like the circular area about the x-axis) and the volume of the solid of revolution is just (*area*)(*distance travelled* by the centre of the area). In our case, *area* = πa^2 and the *distance travelled* is $2\pi b$. Nice, eh?
S: Hey, can I use that instead of all this integration? I'd never be able to do the donut *your* way.
P: Sure, do this one: find the volume when the area enclosed by y = x and $y = x^2$ is revolved about the x-axis.
S: I first draw a picture ... uh, y = x, I know that one, but $y = x^2$ ... that one starts at (0,0) and gets bigger as x gets bigger so I'll just sketch a curve which gets bigger. Then I find the points of intersection; I solve $y = x^2$ and y = x and I get $x^2 = x$ so x = 1. I conclude that ...
P: What! x = 1? Is that all? What about the intersection at x = 0? You can't just go from $x^2 = x$ to x = 1.
S: Sure I can. I divided by x, don't you see? Pay attention. I get ...
P: You can divide by x <u>only</u> if $x \neq 0$ and that means you've just thrown out a solution, namely x = 0. Look. Write $x^2 = x$ as $x^2 - x = 0$ so $x (x - 1) = 0$ so either x = 0 or x = 1. See? I get <u>both</u> solutions.
S: Do you want me to do this, or what? Okay, I get intersections at (0,0) and (1,1). Now I find the area between the curves by using vertical rectangles ... the width is dx and the height ... uh, how do I know which curve is bigger? I mean, which is the upper y-value? I won't worry about that now, I'll just assume $y = x^2$ is bigger so that gives me rectangles with an area of $(x^2 - x)$ dx and I add them all up ... $\int$um them (as you're so fond of saying) ... and I get the area

$\int_0^1 (x^2 - x)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{(2)(3)} = \frac{-1}{6}$. I conclude that the area is $\frac{1}{6}$, so now I ...

P: What! You got a negative answer. So where did you go wrong? I'll tell you where you went wrong, you ...
S: I know, I know ... y = x is bigger, so who cares? I <u>know</u> the area is positive, so I just forget the (-) sign. Pay attention. Now I'm going to find the volume when I revolve this area about the x-axis. I just use (*area*)(*distance travelled* by the centre of the area) and the *area* = 1/6 and *distance travelled* is ... is ... how am I supposed to know where the centre of the area is? I guess it's half-way ... uh, at (1/2,1/2) ... so that'd make a *distance travelled* of about $2\pi(1/2)$ or just π, so the volume is $(1/6)\pi$. How's that?
P: Terrible! You don't even know if x^2 is larger or smaller than x and you're just <u>guessing</u> the location of the centre of the area

and ...

S: Okay, <u>you</u> do it.



P: First I make a *reasonable* diagram (and I know that x^2 is smaller than x when x lies in $0 < x < 1$ since I just have to try one value, say x=.5 then x^2=.25 is smaller). Then I subdivide into vertical rectangles of area $(x - x^2)$ dx and $\int$um from 0 to 1 and get $\frac{1}{6}$ as the area. Next I find the volume by revolving each rectangle about the x-axis. Each gives a "washer" of inner radius $y = x^2$ and outer radius y = x so the volume is $\pi\{ (x)^2 - (x^2)^2\}$ dx and I $\int$um these from x = 0 to x = 1 and get the total volume

$$V = \pi\int_0^1 (x^2 - x^4)dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 =$$

Error!

S: I thought you were going to use (*area*)(*distance travelled* by the centre of the area) to find the volume. But I know why you didn't use it ... you don't know where the centre of the area is, right?

P: Sure I do. It's located at ... uh, the y-coordinate of the centre of area is ... give me a second ... it's $\frac{2}{5}$ (which, by the way, is smaller than your *guess* of $\frac{1}{2}$) .

S: How'd you get that? I mean, how do you know the centre of area is at $y = \frac{2}{5}$?

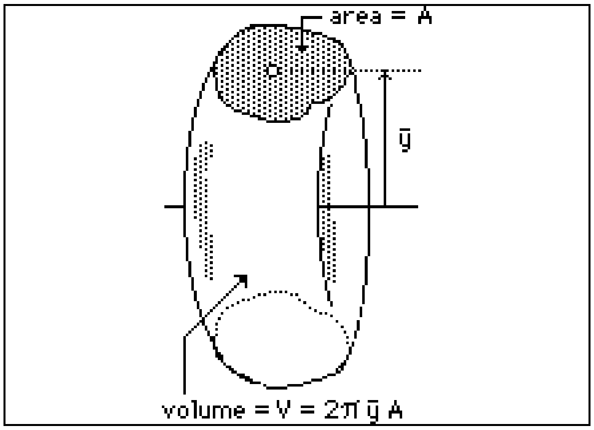
P: Pay attention: I use (*area*)(*distance travelled* by the centre of the area) and I 've computed the area as $\frac{1}{6}$ and the centre of area I take to be at $y = \frac{2}{5}$ so it travels around a circle of radius $\frac{2}{5}$ so I get a volume of (*area*)(*distance travelled*) which is $\frac{1}{6}\left(2\pi\frac{2}{5}\right) = \frac{2\pi}{15}$ and, as we've seen, it's the correct volume. See? $y = \frac{2}{5}$ works!

S: But that's cheating! I bet you just figured out where the centre of area had to be to give the right volume. Am I right?

P: Yes. And that's as good a way as any to <u>define</u> what we mean by the "centre of area". It's located at a point such that (*area*)(*distance travelled* by the centre of the area) gives the volume. For example, the y-coordinate of the centre of area ... let's call it $\bar{y}$ ... satisfies $A\, 2\pi\, \bar{y} = V$ (where A = area and V = volume) so we have a formula for finding the centre of area: $\bar{y} = \frac{V}{2\pi A}$. Nice, eh? You see, it makes sense, this definition of "centre of area", and it works when the area is a circle so that gives us some faith in using this definition and besides, if I <u>define</u> the centre of area in this manner who can argue with me? It's a definition, after all, and we can't argue with a definition! It just has to be a reasonable definition which agrees with what we'd expect the centre of area to be in simple cases. See? Just use $\bar{y} = \frac{V}{2\pi A}$.

S: Can we patent that formula ... or has somebody already discovered it?

P: It's called:



THE THEOREM OF PAPPUS

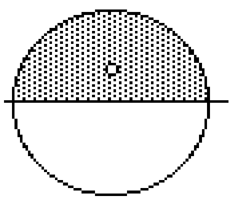
A planar region of area A is revolved about a line
(in the same plane as the region).

The volume of the solid of revolution is given by:

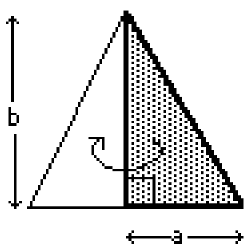
$$\boxed{V = 2\pi \bar{y} A}$$

Given any two of V, $\bar{y}$ and A we can compute the third from $A = 2\pi \bar{y} A$.

Examples:



Since the area of a semi-circle is $A = \frac{1}{2}\pi r^2$ and the volume (when revolved about the diameter) is $V = \frac{4}{3}\pi r^3$, then the centre of area (or CENTROID) is located at $\bar{y} = \frac{V}{2\pi A} = \frac{4}{3\pi} r$ (the distance from the diameter).



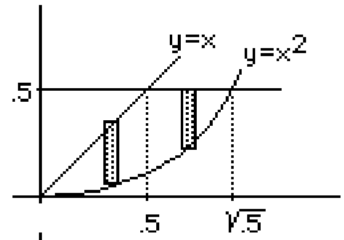
We revolve a right-triangle about a side, generating a right-circular cone. The area of the triangle is $A = \frac{1}{2}$(base) (height) $= \frac{1}{2}$ a b and the volume of the cone is $V = \frac{1}{3}$(area of base) (height) $= \frac{1}{3}\pi a^2 b$, so the CENTROID of the triangle is located a distance $\frac{V}{2\pi A} = \frac{1}{3}$ a from the line about which we revolved.

S: Hey! I knew that! The centroid of a triangle is 1/3 of ... of something. Say, can I use this theorem is you ask me to compute a volume?
P: Only if I say "Use the Theorem of Pappus to compute ..."
S: But why? If it's easier, then I should be allowed to use it!
P: If I want to test you on your ability to set up a definite integral and evaluate it, then I won't allow you to use Pappus. However, by all means use Pappus to test for "reasonableness".
S: But if I get the right answer, surely ...
P: No! Getting the right answer isn't as important as knowing a method of solution. You can write

$\lim_{x\to\infty} \left(\sqrt{1+x} - \sqrt{x}\right) = \infty - \infty = 0$ and get the right answer, but it's not worth any marks because the method is all wrong. On the other hand, even if you get the wrong answer you may *still* get full marks. For example, you solve a complicated problem ... demonstrating your ability to solve this type of problem ... then you end up with (2)(3) which you inadvertently write as "5". I'd give you full marks because I *know* you can multiply "2" times "3" correctly ... I think.
S: Thanks.

Example:

Compute the volume generated when the region enclosed by $y = x$, $y = x^2$ and $y = \frac{1}{2}$ is revolved about the x-axis.

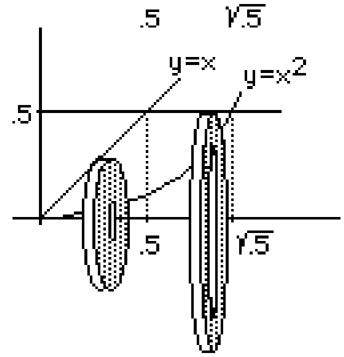


Solution: The points of intersection are (0,0), (.5,.5) and ($\sqrt{.5}$,.5) as shown in the diagram. For $0 \le x \le .5$, the vertical rectangles rise from $y = x^2$ to $y = x$, and in $.5 \le x \le \sqrt{.5}$ they rise from $y = x^2$ to $y = .5$ so we'll need two definite integrals. When we revolve the rectangles in [0,.5] we get "washers" with volume $\pi((x)^2 - (x^2)^2)\,dx$ and when we revolve the rectangles in [.5,$\sqrt{.5}$] the volumes are $\pi((1)^2-(x^2)^2)\,dx$.

The total volume is then:

$$V = \pi\int_0^{.5}(x^2-x^4)\,dx + \pi\int_{.5}^{\sqrt{.5}}(1-x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^{.5} + \pi\left[x - \frac{x^5}{5}\right]_{.5}^{\sqrt{.5}}$$

$= \dfrac{19\pi}{20\sqrt{2}} - \dfrac{11\pi}{24}$ which is roughly $\pi\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\right)$ which is roughly $3(.7 - .5) \approx .6$, and we should check this for reasonableness. However, since we'll be doing this problem again, differently, we'll wait.

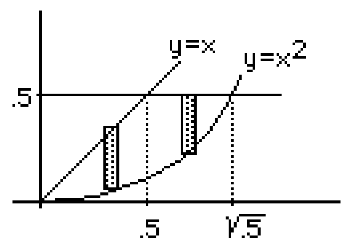


LECTURE 19

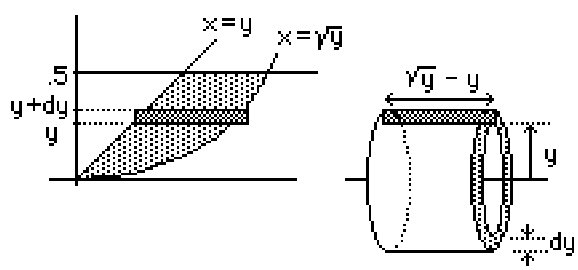
MORE APPLICATIONS OF THE DEFINITE INTEGRAL

Volumes of solids of revolution using horizontal rectangles

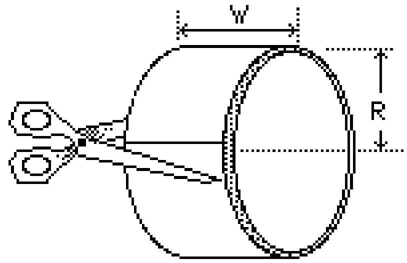
In the previous problem we were faced with a region which had to be subdivided into two parts: $0 \le x \le \frac{1}{2}$ and $\frac{1}{2} \le x \le \frac{1}{\sqrt{2}}$. There were different expression for the height of the vertical rectangles depending upon where the rectangle was located. Had we taken horizontal rectangles (as we often do when computing AREA rather than VOLUME) and revolved each about the x-axis, then we'd get a single definite integral for the volume. That's because a horizontal rectangle, located at the place "y" (between y = 0 and y = .5), *always* runs from $y = x$ (on the left) to $y = x^2$ (on the right).



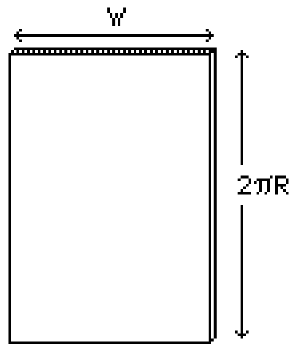
Okay, at the place y the right-most x-value is $x = \sqrt{y}$ and the left-most is $x = y$ so the width of the rectangle is $x_{right} - x_{left} = \sqrt{y} - y$ (compare this to $y_{upper} - y_{lower}$ when we took vertical rectangles!). The height of the horizontal rectangle is dy (it's an increment in "y" rather than an increment in "x", so we call it dy). If we wanted the AREA of the region we'd just $\int$um



via $\displaystyle\int_0^{1/2}(\sqrt{y} - y)\,dy$. However, we want to revolve each horizontal rectangle about the x-axis, giving a VOLUME. The solid generated is a cylindrical shell. (A *very thin* cylindrical shell). We need to know the volume of such a shell (as we needed to know the volume of a "disc" or a "washer" when we took vertical rectangles).



Imagine such a shell as a rectangular piece of paper rolled into a cylinder. To find its volume we cut it, unroll it, and recognize that its volume is just (*width*)(*length*)(*thickness*) = (W)(2πR)(t) where "t" is the thickness and the *length* is just the circumference of a circle of radius R. It's much like painting the sides of a cylindrical soup can, then asking for the volume of paint: the thickness is *very* small.



For our horizontal rectangle we need only identify the *width*, *height* and *thickness*. They're $\sqrt{y} - y$, $2\pi y$ and dy respectively. Then the volume of each cylindrical shell is just $2\pi y\,(\sqrt{y} - y)\,dy$ and the total volume of <u>all</u> such shells, obtained by revolving <u>all</u> horizontal rectangles from y = 0 to y = 1, is the $\int$um , namely:

$$V = 2\pi \int_0^{1/2} y\,(\sqrt{y} - y)\,dy = 2\pi \int_0^{1/2} (y^{3/2} - y^2)\,dy$$

$$= 2\pi \left[\frac{2}{5}\,y^{5/2} - \frac{y^3}{3}\right]_0^{1/2} = \frac{\pi}{5\sqrt{2}} - \frac{\pi}{12} \;.$$

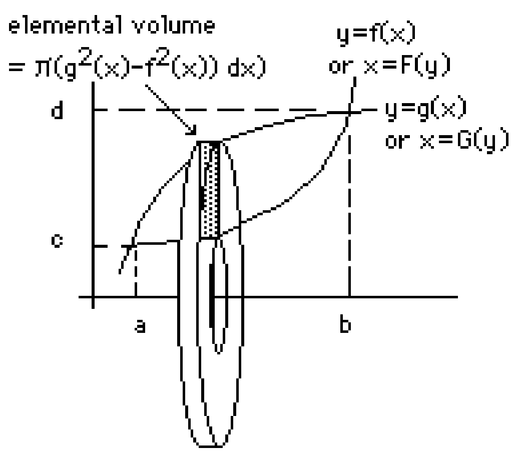
S: Hold on! Last time you got $\frac{19\pi}{20\sqrt{2}} - \frac{11\pi}{24}$!

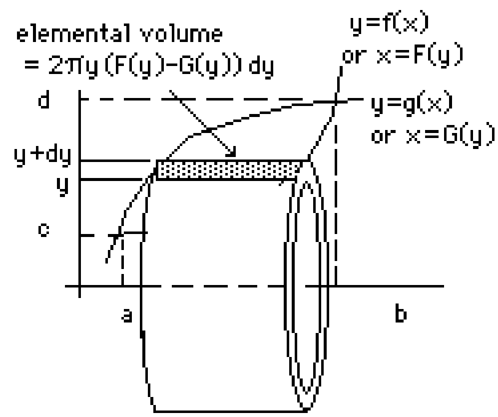
P: Well, I guess they're the same number ... wouldn't you think?

S: But they're <u>not</u> the same! I mean, they sure don't look the same. I mean ...

P: Well look carefully through both solutions and see if you can find any error. That's left as an exercise for the student ... and guess who's the student?

When we use horizontal rectangles we need to solve for x, from the equations for the curves which enclose the region. That's because we need the width of the rectangle located at some y-value and that's $x_{right} - x_{left}$. In general, if the region we're revolving is bounded by, say y = f(x) and y = g(x) as shown below, then, for <u>vertical</u> rectangles we'd get a thin washer with volume $\pi(g^2(x) - f^2(x))\,dx$ (assuming that y = g(x) is the upper curve) and for horizontal rectangles we'd get a thin cylindrical shell with volume $2\pi y\,(F(y) - G(y))\,dy$ where y = f(x), when solved for x gives x = F(y) and y = g(x) gives x = G(y).





NOTE: F and f are inverse functions ... and g and G are also inverse functions ... and it's not always clear which will give the simpler integrals to evaluate: <u>horizontal</u> or <u>vertical</u> rectangles.

Example:

Compute the volume when the region enclosed by y = sin x, y = x and y = 1 is revolved about the x-axis.
Solution: If we take <u>vertical</u> rectangles we'd get the volume expressed as

$V = \pi\int_0^1 (x^2 - \sin^2 x)\,dx + \pi\int_1^{\pi/2} (1 - \sin^2 x)\,dx$ both of which are pretty easy (the only tough integral is $\int \sin^2 x\,dx$ but if we write $\sin^2 x = \frac{1 - \cos 2x}{2}$ it's easy) ... BUT there <u>are</u> two integrals. If we try <u>horizontal</u> rectangles we solve y = x for x = y (pretty easy) and y = sin x for x = arcsin y (and right away we expect a problem!). The volume (expressed in terms of horizontal rectangles, revolved) is $V = 2\pi\int_0^1 y\,(\arcsin y - y)\,dy$ which, although it's only a single integral, is more difficult (because of the arcsine function).

S: Aren't there any *real* problems we can ... uh, *you* can solve? I mean, problems which aren't just math problems?
P: Using definite integrals? Sure. Lots of them. Remember, the definite integral can represent things other than AREA or VOLUME. This area/volume stuff is just so we can picture what's happening.

Example: The speed of an object depends upon the time according to $v(t) = t^2 + t$ metres/second, where t is measured in seconds. Determine the distance travelled during the time interval from t = 1 to t = 3.

Solution: If x(t) is the distance travelled in a time t, then $\frac{dx}{dt} = v = t^2 + t$ so $x(t) = \int (t^2+t)dt = \frac{t^3}{3} + \frac{t^2}{2} + C$.

Further, x = 0 when t = 0 so (substituting) we find 0 = 0 + C so C = 0 and $x(t) = \frac{t^3}{3} + \frac{t^2}{2}$. When t = 1,

$x(1) = \frac{1^3}{3} + \frac{1^2}{2}$ and when t = 3, $x(3) = \frac{3^3}{3} + \frac{3^2}{2}$. The distance travelled during the time from t = 1 to t = 3 is

$x(3) - x(1) = \left(\frac{3^3}{3} + \frac{3^2}{2}\right) - \left(\frac{1^3}{3} + \frac{1^2}{2}\right) = \left[\frac{t^3}{3} + \frac{t^2}{2}\right]_1^3$ and this is just $\int_1^3 (t^2 + t)\,dt$. We conclude that the distance can be found directly by integrating the speed from t = 1 to t = 3 ... a definite integral: $\int_1^3 v(t)\,dt$.

Note that v(t) is the rate of change of distance measured in, say, *km/hour*, and x(t) is the distance travelled (in *km*) after a time t. Hence v(t) dt is the number of *kilometres* during dt *hours* and $\int$umming gives the total number of *kilometres*. This understanding goes with a number of similar problems.

- If A(z) is the change in the number of *acres per foot*, at z *feet*, then A(z) dz is the number of *acres* for a change of dz *feet* and $\int_a^b A(z)\,dz$ is the total change in the number of *acres* from z = a to z = b *feet*.
- If U(T) is the change in *pressure per degree* Celsius, when the temperature is T°, then U(T) dT is the pressure change when the temperature changes from T to T+dT and $\int$umming gives $\int_a^b U(T)\,dT$ as the total change in pressure when the temperature changes from T = a to T = b.
- If the growth is M(t) *cm per month*, after t *months*, then M(t) dt is the number of *centimetres* of growth during the time from t to t+dt and $\int_a^b M(t)\,dt$ is the total growth from t = a to t = b *months*.

In general, if W is a rate of change, measured in *whatzits* per *doodle*, and if D, the amount of *doodles* changes from D = a to D = b, then the total change is W $\frac{whatzits}{doodle}$ x (b - a) *doodles* = W (b - a) *whatzits*. Unfortunately, if the rate of change, W, depends upon the number of *doodles*, say W = f(D), then we assume that D changes in microscopic steps during which W = f(D) is essentially constant ... during each microscopic step. When the amount of *doodles* changes from D to D+dD, the change in *whatzits* is f(D) dD. The total change is then obtained by $\int$umming all these microscopic changes: $\int_a^b f(D)\, dD$. Note that if f(D) is constant at the value W, the integration yields W (b - a) as expected.

Example: In drilling a well, the cost per metre depends upon the type of sand, gravel or rock which must be excavated. Suppose the cost is C(x) *dollars/metre* at a depth x *metres*. (As x changes, the type of material changes, hence the cost changes.) Express, as a definite integral, the cost in digging a well of H *metres*.
Solution: In digging from x to x+dx, the cost per metre is C(x) so the cost for just this short piece is C(x) dx dollars. The total cost, for H metres, is $\int_0^H C(x)\, dx$ dollars.

S: I don't understand this at all.
P: Patience.

Example: The force required to extend a spring a distance of x cm is F(x) Newtons. (The more the extension, the greater the force required ... and we'll measure the force in "Newtons".) If WORK is measured as (*force*)(*distance*), how much work is done in extending the spring a distance h metres.
Solution: In extending from x to x+dx, the force is F(x) and the work is F(x) dx. The total work in extending from x = 0 to x = h is $\int_0^h F(x)\, dx$ (measured in a strange unit: *Newton-centimetres*!)

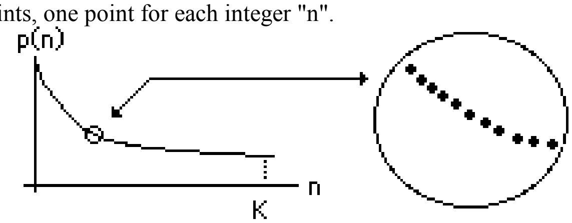
S: I still don't ...
P: Okay, here's how it goes. We're digging a well and we're at a depth of x metres. At this depth the cost is $C for every metre, so digging a small, incremental distance dx will cost $C dx. The next small step of length dx will also cost $C dx , but C changes, so we just $\int$um the costs via $\int_0^H C(x)\, dx$. See? The definite integral does it all for us. It's like finding the area under a curve y = f(x). We consider only the part from x to x+dx and assume that f(x) is constant there, at the value f(x). The area under the curve is then approximated by f(x) dx (i.e. the area of a vertical rectangle). When we $\int$um from x = a to x = b we get the <u>exact</u> area, namely $\int_a^b f(x)\, dx$. See how it works? Consider extending the spring. We do it in small steps of length dx (like digging the well in small steps of length dx). When we're at a length x the force is F(x) and even though it changes with x we assume it's constant over the small interval dx and approximate the work by (*force*)(*distance*) = f(x) dx. Now $\int$um all such incremental work, from x = 0 to h and get WORK = $\int_0^h f(x)\, dx$ and it's <u>exact</u>.
S: Seems fishy that you can write down an approximation then, poof, you get the exact answer.
P: Go back and review Riemann SUMS and taking the limit as h–>0 and n–>∞ and all that good stuff. It's this limiting procedure which gives us the exact answer. Remember? The errors actually go to zero. That's why I never say "we sum this or that" I always say "we $\int$um this or that". Buried in that expression is a world of ingenuity. Let's do the well-digging or the work problem by constructing a Riemann SUM then taking the limit. Are you interested?
S: Will it be on the final exam?
P: Forget it. We'll go on with another application.

Example: The cost of manufacturing an item depends upon the number of items manufactured. For the first few items the cost is high and the profit low, but as we produce more items the cost decreases hence the profits (when we sell the items) increase. Suppose the profit for the n^{th} item is p(n) dollars per item. Express, as a definite integral, the profit in producing (and selling) K items.

Solution: If we've produced n items and produce just a few more items, say dn items, then the profit is approximately p(n) dn. Now we $\int$um the profits for all K items and get the total profit as $\int_1^K p(n)\,dn$.

S: Come on! "We produce just a few more items, like maybe *dn* more"? I thought dx and dy and dn were supposed to be really small. Are you talking about 1/1000 or an item or what?

P: That's a good question. In fact, a very good question. How come you never asked when we maximized the yield of apples from our orchard? Remember? We asked "How many more trees should be added to maximize the yield". I invented the problem so there would be an integer number of trees to add. You should have asked "What if we had to add 1/1000 of a tree, what then?" You see, if we are to apply the methods of calculus we have to assume that the independent variable can take on <u>any</u> value (in some domain). Then we can talk about "dx" and so on. If the independent variable is the *number of trees* or the *number of items* then it's clearly an integer so we fudge things a bit and consider it a "continuous variable". When we get an answer like "add 57.8 trees" we just add 58 more trees. See? Actually, if our manufacturing plant produced tens of thousands of items the graph of p(n) versus n might look like the the diagram below ... and if we expand a piece we'd see that p(n) is a bunch of points, one point for each integer "n".



In fact, the actual total profit is really "the profit for the first item" + "the profit for the second" and so on. In other words, it's p(1) + p(2) + ... + p(K). However, it's often easier to use $\int_1^K p(n)\,dn$ and recognize that it's a reasonable approximation if n is large.

S: Aha! Approximation! I <u>said</u> your $\int$ummimg just gave approximations, didn't I?

P: Okay, in this case we *do* get an approximation, but not because of the definite integral. If we have an equation for p(n), such as $p(n) = 5 + \frac{1}{n}$, then $\int_0^K p(n)\,dn = \left[5n + ln\ n\right]_0^K = 5K + ln\ K$ and this would be <u>exactly</u> the area under the curve and <u>exactly</u> the total profit if n were a continuous variable. The mathematics can't tell that "n" is an integer; it thinks it's *any* real number. It's <u>we</u> who have introduced the error (fortunately, a small error in many cases of interest) by resorting to the methods of calculus.

S: No, it's not <u>we</u>. I didn't introduce anything. It's <u>you</u> who introduced the error ... and here's another one for you. Your graph of p(n) is *decreasing* and you said the profit would *increase*. Ha!

P: Pay attention. Here's another application.

S: Wait. You said it's easier to use $\int_1^K p(n)\,dn$ than to use p(1) + p(2) + ... + p(K). You're kidding, right? I mean, integrating is easier than adding?

P: Okay, let's see <u>you</u> add p(1) + p(2) + ... when $p(n) = 5 + \frac{1}{n}$. You'd get $(5+\frac{1}{1}) + (5+\frac{1}{2}) + (5+\frac{1}{3}) + ... + (5+\frac{1}{K})$ and what's *that* add up to? There are 5's which add up to 5K, by what about $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{K}$? That's not easy. In fact (and this may surprise you), if K is large, it adds up to *ln* K very nearly.

S: Nothing surprises me ... and what's "very nearly".

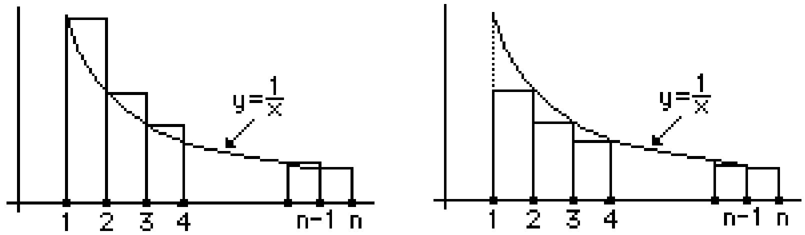
P: Remember when we talked about one thing being "close to" another? We agreed that the way to compare them is to take their ratio and see if it's very close to 1. In this case, the ratio $\frac{1/1 + 1/2 + 1/3 + ... + 1/K}{ln\ K}$ <u>is</u> very close to 1, when K is large.

In fact, you may be interested to know that $\lim_{K \to \infty} \frac{1/1 + 1/2 + 1/3 + ... + 1/K}{ln\ K} = 1$.

S: I'm not.

P: Pay attention.

Consider the graph of $y = \frac{1}{x}$ from x = 1 to x = n. We divide it up into rectangles as shown.



First we consider n rectangles whose total area is larger than the area under the curve, then rectangles whose total area is smaller than the area under the curve. In each case the width of each rectangle is "1", but the heights differ. In the first case, the heights (given by $y = \frac{1}{x}$) are $\frac{1}{1}$ and $\frac{1}{2}$ and $\frac{1}{3}$ etc. so the total area of these rectangles is just: $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1}$ (and this is larger than the area under the curve $y = \frac{1}{x}$) . The second set of rectangles have heights $\frac{1}{2}$ and $\frac{1}{3}$ and $\frac{1}{4}$ etc. so the total area of rectangles is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$ and this is smaller than the area under the curve. But we know exactly the area under the curve; it's $\int_1^n \frac{dx}{x} = \ln n$ so (using the smaller rectangles):

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} < \ln n \qquad \text{hence} \qquad 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} < 1 + \ln n \ \text{ hence}$$

$$\frac{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}}{\ln n} < \frac{1}{\ln n} + 1 \qquad \text{hence} \qquad \lim_{n\to\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}}{\ln n} \le 1.$$

Now use the larger rectangles and get:

$$\ln n < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} \qquad \text{hence} \qquad \frac{1}{n} + \ln n < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \qquad \text{hence}$$

$$\frac{1}{n \ln n} + 1 < \frac{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}}{\ln n} \qquad \text{hence} \qquad 1 \le \lim_{n\to\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}}{\ln n} .$$

P: Well? What's our conclusion? Did you notice that we used the ol' SQUEEZE again?
S: zzzzzz
P: Well, at least you see that the definite integral can be used for lots of things, not just AREAs. We'll do some more:

AVERAGE VALUE OF A FUNCTION:

Suppose we wish to determine the average value of a function. To be specific, suppose the temperature during a 24 hour period varies according to $T = 20 + 10 \sin \frac{\pi t}{12}$ so it starts out (at t = 0) at 20˚C and varies from a minimum of 10˚C to a maximum of 30˚C, then starts all over again, 24 hours later. What's the average temperature? Clearly we'd want to measure the temperature many times throughout the day and average them. Suppose we measure the temperature n times and get $T_1, T_2, ...,T_n$. Then the average would be $\frac{T_1 + T_2 + T_3 + ... + T_n}{n}$. To be more accurate we should measure T every second or perhaps every 1/10 second, etc. What would we get if we measured T an infinite number of times? If we could do that, we'd have the average temperature exactly.

We subdivide the 24 hour time interval into n subintervals of length $h = \frac{24}{n}$ (does that sound familiar?) and measure the temperature at times t = h, 2h, 3h, ... , nh (which, of course, is the same as t = 24). The average is then $\frac{T(h)+T(2h)+T(3h)+...+T(nh)}{n}$ and we'd like to compute the limit of this as $n\to\infty$. To this end we rewrite it as:

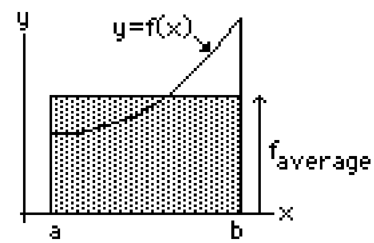
$$\frac{T(h)h+T(2h)h+T(3h)h+...+T(nh)h}{nh} = \frac{1}{24}\sum_{k=1}^{24} T(kh)\,h$$ where we've multiplied numerator and denominator by h and put nh = 24. Now we let n–>∞ and recognize the limit of a Riemann SUM which yields: $\frac{1}{24}\int_0^{24} T(t)\,dt$. For our particular $T = 20 + 10 \sin\frac{\pi t}{12}$ we'd get an average temperature of $\frac{1}{24}\left[20t - \frac{120}{\pi}\cos\frac{\pi t}{12}\right]_0^{24}$ = 20°C which is not too surprising, I guess.

We generalize this:

$$\boxed{\text{The average value of } f(x) \text{ over the interval } x = a \text{ to } x = b \text{ is } \frac{1}{b-a}\int_a^b f(x)\,dx}$$

One interesting thing about this "average" value is the following:

Suppose y = f(x) is positive on [a,b] and we graph it. The "area under f(x)" would be (*width*)(*height*) provided f(x) were a constant (i.e. the graph was a horizontal line, so f(x) = constant). If f(x) weren't a constant, what *height* should we use in order to get the correct value for the area? In other words, using *width* = b - a, what *height* would make the area under the curve equal to the rectangular area: (*width*)(*height*) = (b - a) (*height*). Well, since the area is



$\int_a^b f(x)\,dx$, we'd want (b - a) (*height*) = $\int_a^b f(x)\,dx$ so that the correct *height* is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

It's just the "average height" as given by our formula for "average"... so we think it's a good definition of "average".

Example: The velocity of a vehicle is given by $v(t) = 50 + 20e^{-t}$ km/hour. Calculate its average velocity over the time interval from t = 0 to t = 3 hours.

Solution:

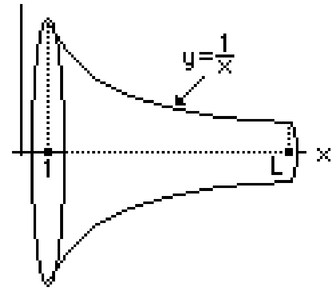
$$v_{average} = \frac{1}{3-0}\int_0^3 (50+20e^{-t})\,dt = \frac{1}{3}\left[50t - 20e^{-t}\right]_0^3 = \frac{1}{3}\left\{ \left(50(3) - 20e^{-3}\right) - \left(59(0) - 20e^{-0}\right)\right\} = \frac{170 - 20e^{-3}}{3} \text{ km/hour.}$$

Note that $\int_{t_1}^{t_2} v(t)\,dt$ is the total distance travelled by the vehicle during the time interval from $t = t_1$ to $t = t_2$, so

$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{\text{total distance}}{\text{total time}}$ is just what we'd expect the average velocity to be!

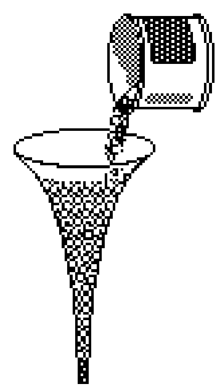
A PARADOX:

Consider the volume when the area under $y = \frac{1}{x}$ is revolved about the x-axis, from x = 1 to x = L. We have (using vertical rectangles) $V = \pi \int_1^L \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^L = \pi\left(1 - \frac{1}{L}\right)$. Now consider the area under the curve, namely $A = \int_1^L \frac{1}{x}\,dx = [\ln x]_1^L = \ln L$.



Notice that revolving the curve $y = \frac{1}{x}$ about the x-axis generates a kind of container which, when filled with paint, would require $\pi\left(1 - \frac{1}{L}\right)$ metres^3 of paint (assuming both x and y are measured in metres). As L–>∞ we'd never need more than π m^3 of paint ... even though the container becomes infinite in length. Now look at the area: *ln* L–>∞ as L–>∞. That means that even though the infinite container is filled with paint (π m^3 worth of paint), it's not enough paint to cover this infinite area (which, after all, lies entirely within the container!).

S: Do you expect me to believe that?
P: Do you see anything wrong with my reasoning?
S: I guess not ... but I still don't believe it. I mean, is the math is so stupid that ...
P: It's never the math that's stupid, it's us.
S: Speak for yourself.
P: I bet you think it always takes more and more paint to cover bigger and bigger areas. Pick a big area.
S: I pick 10^{10} metres^2.
P: Okay, I cover it with paint 10^{-30} metres thick and that'll take (area)(thickness) = $10^{10}\ 10^{-30} = 10^{-20}$ metres^3 of paint. Not much, eh? And if you picked a larger area I'd just use a much thinner coat of paint and could paint your larger area with even <u>less</u> paint!
S: So?
P: So let's talk about *real* paint with molecules having a radius of ... what's a good symbol for a small radius?
S: How about ε? I always liked ε. He's small but he gets around and ...
P: Pay attention. We'll fill our "infinite" container with paint having molecules which are ε metres in radius. That means the paint will only fill the container until the radius of the neck of the container is $y = \frac{1}{L} = \varepsilon$. That means $L = \frac{1}{\varepsilon}$ in our formula for volume, so the volume is $V = \pi(1 - \varepsilon)$ metres^3. Now our area is $\ln L = \ln\frac{1}{\varepsilon}$ and covered with paint of molecular radius ε will give a thickness 2ε and hence a volume of: (area)(thickness) = $2\varepsilon \ln\frac{1}{\varepsilon}$ metres^3. Did you know that $2\varepsilon \ln\frac{1}{\varepsilon}$ is <u>always</u> less than $\pi(1 - \varepsilon)$? That means there is <u>always</u> enough paint in the container to cover the area. What do you think of that? You see, comparing an area with a volume is wrong, wrong wrong! They're don't even have the same dimensions. It's like comparing apples to oranges. There really is no paradox.



S: Do I have to know this for the final ...

LECTURE 20

IMPROPER INTEGRALS

At the end of the last lecture we considered $\int_1^L \frac{dx}{x}$, the area under $y = \frac{1}{x}$ from x = 1 to x = L ... then we let $L \to \infty$. It would be tempting to write the infinite area as $\int_1^\infty \frac{dx}{x}$, but then we'd recall the definition of the "definite integral" $\int_a^b f(x)dx$ and realize that this collection of symbols means the limit, as $n \to \infty$, of a Riemann SUM of n rectangular areas each of the form $f(x_i)\,h$, where x_i is some point in the i^{th} subinterval (so $f(x_i)$ gives the height of the rectangle) and h is the width of each subinterval (hence the width of each rectangular area), and, for $\int_a^b f(x)dx$, we have $h = \frac{b-a}{n}$. Now what happens if $b = \infty$, as in $\int_1^\infty \frac{1}{x}dx$? Is $h = \frac{\infty - 1}{n}$? Certainly not. How then do we subdivide the infinite interval $1 \le x < \infty$ so that we have n rectangles each with a finite (not infinite!) area??!%$#*

S: What's wrong with an infinite area? That's what it is, right? I mean, when you did this problem you said the area is infinite, so what's wrong with ...

P: Okay, you have a point there. So let me consider the volume of revolution when the above area is revolved about the x-axis, namely $\pi\int_1^L \frac{dx}{x^2}$. Now, as $L \to \infty$, we actually get a finite volume, yet if we look back at the definition of the definite integral ... as a limit of a Riemann SUM ... we'd run into the same problem if we let $L \to \infty$ and wrote the volume as $\pi\int_1^\infty \frac{dx}{x^2}$. The interval is infinite in extent hence if we subdivide into n subintervals (say 100 or 1000 subintervals), we'd have rectangles of infinite width.

S: But you already got an answer for that problem. You said ... if I remember correctly ...
$V = \pi\int_1^L \frac{1}{x^2}dx = \pi\left[-\frac{1}{x}\right]_1^L = \pi\left(1 - \frac{1}{L}\right)$ then you let $L \to \infty$ and got the answer $V = \pi$. What's wrong with that? No Riemann SUM, no subintervals, no infinite widths, no infinite anything!

P: You know, you're becoming very clever. That's *exactly* what we'll do with an integral where the upper limit is ∞.

The collection of symbols $\int_1^\infty f(x)\,dx$ means the limit of a Riemann SUM ... that's the definition ... we have no choice ... and that's where we have a problem because each term in the Riemann SUM is infinite! However, if we do as suggested, we avoid this problem. Hence, we will DEFINE (and this is important!) this type of so-called IMPROPER INTEGRAL as follows:

DEFINITION of an IMPROPER INTEGRAL

$$\int_a^\infty f(x)\,dx = \lim_{L\to\infty} \int_a^L f(x)\,dx$$

provided the limit exists

In other words, if somebody gives you an integral such as $\pi \int_1^\infty \frac{1}{x^2}\,dx$, you just evaluate $\pi \int_1^L \frac{1}{x^2}\,dx$ which is a perfectly *proper* integral (nothing infinite anywhere) ... and you have a warm feeling all over because $\pi \int_1^L \frac{1}{x^2}\,dx$ has a perfectly acceptable definition in terms the limit of a Riemann SUM, with subintervals of perfectly acceptable width $h = \frac{L-1}{n}$ ... then, having evaluated $\pi \int_1^L \frac{1}{x^2}\,dx$, you let $L\to\infty$! If you get a limit (as is the case here), you can say " $\pi \int_1^\infty \frac{1}{x^2}\,dx$ converges to π ", or simply $\pi \int_1^\infty \frac{1}{x^2}\,dx = \pi$.

S: What's all this "converges to π" stuff? I mean, ...

P: I admit it seems a little unusual to introduce this phrase, but it'll become clearer in the next calculus course. In the meantime, to impress your friends, you can say $\int_1^\infty \frac{dx}{x}$ diverges but $\int_1^\infty \frac{dx}{x^2}$ converges, and I might ask you to see if a certain "improper integral" converges or diverges.

Example: Which of the following improper integrals converge?

(a) $\int_0^\infty \frac{dx}{1+x}$ (b) $\int_0^\infty e^{-x}\,dx$ (c) $\int_{-1}^\infty \sin x\,dx$ (d) $\int_{-\infty}^\infty e^{-x}\,dx$

Solutions:

(a) By definition, $\int_0^\infty \frac{dx}{1+x} = \lim_{L\to\infty} \int_0^L \frac{dx}{1+x} = \lim_{L\to\infty} \left[ln\,(1+x)\right]_1^L = \lim_{L\to\infty} \left(ln(1+L) - ln\,1\right) = \lim_{L\to\infty} (ln\,(1+L)) = \infty$, hence $\int_0^\infty \frac{dx}{1+x}$ diverges.

(b) $\int_0^\infty e^{-x}\,dx = \lim_{L\to\infty} \int_0^\infty e^{-x} dx = \lim_{L\to\infty} [-e^{-x}]_0^L = \lim_{L\to\infty} (1 - e^{-L}) = 1$, so $\int_0^\infty e^{-x}\,dx$ converges to 1.

(c) $\int_{-1}^\infty \sin x\,dx = \lim_{L\to\infty} [-\cos x]_{-1}^L = \lim_{L\to\infty} (\cos(-1) - \cos L)$ but $\cos L$ has no limit, so $\int_{-1}^\infty \sin x\,dx$ doesn't converge (hence, it diverges).

(d) Here we write $\int_{-\infty}^\infty e^{-x}\,dx = \int_{-\infty}^0 e^{-x}\,dx + \int_0^\infty e^{-x}\,dx$ and consider each integral separately. We know that $\int_0^\infty e^{-x}\,dx$ converges to 1 (as seen above, in part (b)), so we investigate $\int_{-\infty}^0 e^{-x}\,dx = \lim_{L\to-\infty} \int_L^0 e^{-x}\,dx =$ $\lim_{L\to-\infty} [-e^{-x}]_L^0 = \lim_{L\to-\infty} (-1 + e^{-L}) = \infty$, so $\int_{-\infty}^0 e^{-x}\,dx$, and consequently $\int_{-\infty}^\infty e^{-x}\,dx$, diverges.

S: Hold on! First, why do you split $\int_{-\infty}^\infty$ as $\int_{-\infty}^0 + \int_0^\infty$? Why not $\int_{-\infty}^\infty = \int_{-\infty}^1 + \int_1^\infty$? Second, ...

P: First things first. I could certainly have split the infinite interval, $-\infty < x < \infty$, at $x = 1$ rather than at $x = 0$. I'd still find that $\int_{-\infty}^\infty e^{-x}\,dx$ diverges. It really doesn't matter where I split it. Not only that, I could have left the integral in one piece and considered $\int_A^B f(x)\,dx$ then let $A \to -\infty$ (that'd give me $\int_{-\infty}^B f(x)\,dx$) then let $B\to\infty$ (to get $\int_{-\infty}^\infty e^{-x}\,dx$). See?

S: No.

P: Okay, since we already have a definition for an integral where <u>one</u> limit is infinite, let's agree to always split at zero when we have an integral from $-\infty$ to ∞, then use the previous definition on each improper integral:

$$\boxed{\int_{-\infty}^\infty f(x)\,dx = \lim_{A\to-\infty} \int_A^0 f(x)\,dx + \lim_{B\to\infty} \int_0^B f(x)\,dx}$$

S: I can't really believe that such animals ever, ever show up in a *real* problem. I mean, integrals which go from $-\infty$ to ∞. I mean ...

P: Oh, they do. If f(x) is a probability density, meaning that f(x) dx is the probability that a certain random variable lies between x and x + dx, then $\int_A^B f(x)\,dx$ is the probability that the variable lies between A and B and if all values of the variable in $-\infty$ to ∞ are possible (with perhaps different probabilities), then $\int_{-\infty}^\infty f(x)\,dx = 1$, meaning that there is a 100% chance of the variable being in $-\infty$ to ∞ ... and all probability density functions, f(x), must satisfy this constraint, namely that $\int_{-\infty}^\infty f(x)\,dx = 1$. For example, the famous "normal" or Gaussian probability density, $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$, describes (among other things) the distribution of random errors in a measurement and satisfies

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1$ (as it must). In fact, $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ would be the probability that the random variable lies between -1 and 1. In fact ...

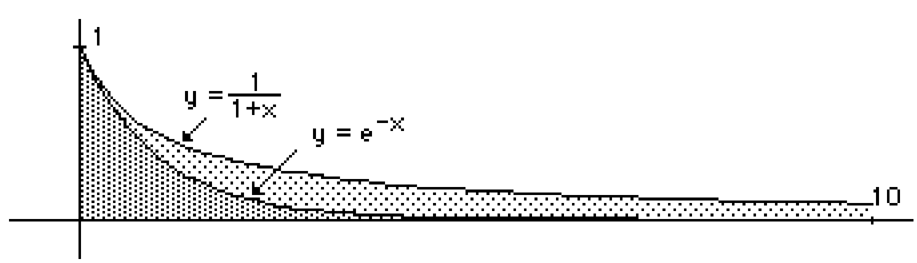
S: Wait! I haven't the foggiest idea of what you're talking about.

P: Sorry ... got carried away ... but it doesn't matter as long as you understand that improper integrals <u>do</u> arise in real problems. Do you?

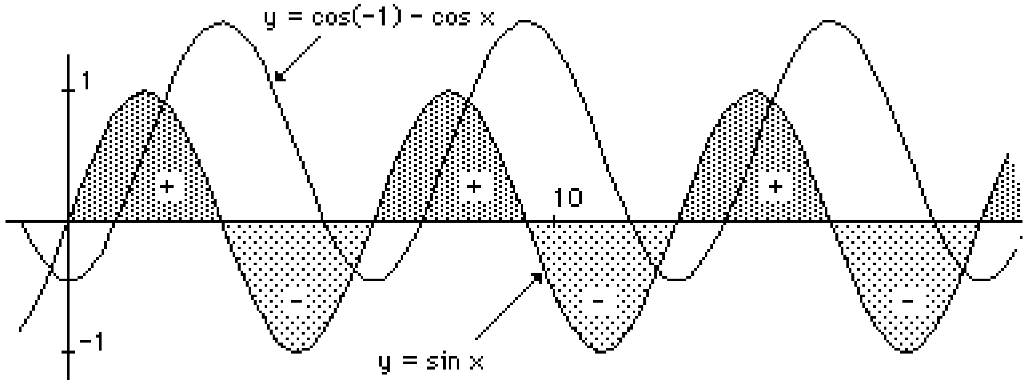
S: I guess so.

NOTE: $\int_a^{\infty} f(x)\,dx$ is defined as the limit of $\int_a^{L} f(x)\,dx$ as $L \to \infty$, and if $f(x) \geq 0$ we can interpret $\int_a^{L} f(x)\,dx$ as the area "under y = f(x) from x = a to x = L". Clearly, if $\int_a^{\infty} f(x)\,dx$ is to converge, the area $\int_a^{L} f(x)\,dx$ must remain finite as $L \to \infty$ and this allows us ... sometimes ... to see clearly whether $\int_a^{\infty} f(x)\,dx$ converges. For example, consider $\int_1^{\infty} e^{-x^2}dx = \lim_{L\to\infty} \int_1^{L} e^{-x^2}dx$. As L increases, so does $\int_1^{L} e^{-x^2}dx$ (since the area under $y = e^{-x^2}$ increases) . To show that $\int_1^{\infty} e^{-x^2}dx$ converges, we need only show that the area $\int_1^{L} e^{-x^2}dx$ doesn't become infinite as $L \to \infty$. But $e^{-x^2} \leq e^{-x}$ when $x \geq 1$ (since $x^2 \geq x$ when $x \geq 1$) so the area under $y = e^{-x^2}$ is less than the area under $y = e^{-x}$ (from x = 1 to x = L) and this latter area is $\int_1^{L} e^{-x}\,dx = \left[- e^{-x}\right]_1^{L} = e^{-1} - e^{-L} = \frac{1}{e} - \frac{1}{e^L}$ which is less than $\frac{1}{e}$. We conclude that $\int_1^{L} e^{-x^2}dx$ is always less than $\frac{1}{e}$, hence cannot become infinite as $L \to \infty$ ... so $\int_1^{\infty} e^{-x^2}dx$ converges ... and we showed this without having to evaluate the integral $\int_1^{L} e^{-x^2}dx$ (which would be tough!) .

In fact, if we stare at the "area under y = f(x) from x = a to x = L", it seems clear that, in order for $\int_a^{\infty} f(x)\,dx$ to converge, the curve y = f(x) must approach zero very rapidly.



We've already seen in example (a) above, that $\int_0^{\infty} \frac{dx}{1+x}$ diverges and that's because $y = \frac{1}{1+x}$, although it *does* approach zero, doesn't do it quickly enough. On the other hand, because of the explosive growth of e^x, $y = e^{-x} = \frac{1}{e^x}$ $\to 0$ very rapidly ... and we saw in example (b) that $\int_0^\infty e^{-x}\,dx$, in fact, converges.

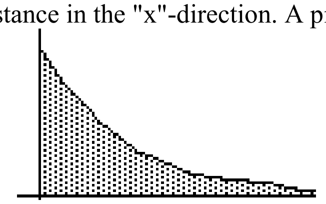


On the other hand, in example (c), y = sin x oscillates continually between -1 and +1 and the "area" $\int_{-1}^{L} \sin x\,dx = \cos(-1) - \cos L$ oscillates between a minimum of cos (-1) -1 and a maximum of cos(-1) + 1 and never approaches any limiting value. On the other hand ...

S: How many hands have we got here? Can't we just do more problems?

Another Kind of Improper Integral

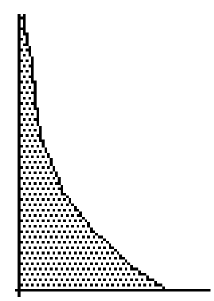
If the limits of integration (in $\int_a^b f(x)\,dx$) are infinite ($a = -\infty$ or $b = \infty$ ... or both) we have an IMPROPER INTEGRAL and we now have a mechanism for dealing with these integrals. If $a = -\infty$ and/or $b = \infty$ it means the extent of variation of "x" is infinite. It means the graph extends an infinite distance in the "x"-direction. A picture is worth a thousand words, so look at the picture ===>>



Here we consider an area of infinite extent in the "x"-direction, from say x = 0 to x = ∞. (I keep putting "x" in quotes for a reason!).

Now we rotate the area shown (actually, we interchange x and y), so that it's infinite in extent in the "y"-direction!

Here's the picture ===>>



Remember, it's the same area! If the first is finite, then so is the second. If the first is infinite, so is the second. However, we now have a different kind of problem. For this second area, it will have a form such as $\int_0^1 f(x)\,dx$ where y = f(x) has a vertical asymptote at x = 0, that is $f(x) \to \infty$ as $x \to 0^+$. Such an integral as this cannot be defined in terms of a limit of a Riemann SUM. Why? Because now the elemental rectangles can have infinite *height*! (Previously, they had infinite *width*.) However, we avoid the problem (of not being able to define these integrals in terms of a limit of Riemann SUMs) by defining them as follows:

$$\text{If } f(x) \text{ is discontinuous at } x = a, \text{ then} \int_a^b f(x)\,dx = \lim_{L \to a^+} \int_L^b f(x)\,dx$$

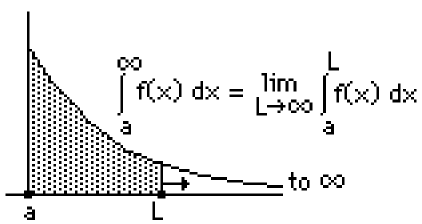
In other words, we avoid integrating from x = a and instead begin at x = L (where L > a). This avoids the place where f(x) is discontinuous and the integral is well-defined and we're happy. Then we let L sneak up on x = a (from values greater than a, of course, hence $L \to a^+$) and pray there is a limiting value. If so, that's the value of $\int_a^b f(x)dx$. If not, then this improper integral doesn't <u>have</u> a value! (i.e it diverges).

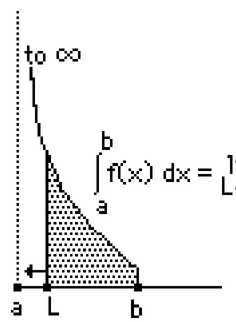
Example: Evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$, if possible.

Solution: This is an improper integral (since $f(x) = \frac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$) so we consider $\int_L^1 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_L^1 = 2 - 2\sqrt{L}$ and let $L \to 0^+$ to get $\int_0^1 \frac{dx}{\sqrt{x}} = 2$. The "area under $y = \frac{1}{\sqrt{x}}$ from x = 0 to x = 1" is infinite in extent (in the "y"-direction), nevertheless it has a finite value.

We illustrate, *graphically*, what we do with these two types of improper integrals. Do you know why?

S: Yeah. A picture is worth a thousand words.





Remember, for $\int_a^\infty f(x)\,dx$ to have a value (i.e to "exist" or to "converge"), f(x) must get small enough fast enough. We've already seen that $y = \frac{1}{x}$ doesn't get small enough fast enough, because $\int_1^\infty \frac{dx}{x}$ diverges (to infinity). (We saw that when we considered the area under $y = \frac{1}{x}$, remember?) On the other hand, the volume of revolution gave rise to an integral $\int_1^\infty \frac{dx}{x^2}$ which <u>did</u> converge. What does that tell you?

S: Huh?

P: Pay attention. It means that $f(x) = \frac{1}{x}$ doesn't get small enough fast enough, but $f(x) = \frac{1}{x^2}$ does! Now, a problem for *you*. If

$f(x) = \frac{1}{x^p}$, how big do we have to make the number p in order that $\int_1^\infty \frac{dx}{x^p}$ converge?

S: Well, that's pretty easy. If p = 2, it converges and if p = 1 it doesn't.

P: And if p = 1.5 ?

S: Oh, yeah, well ... we write $\int_1^\infty \frac{dx}{x^p} = \left[\frac{1}{-p+1} x^{-p+1} \right]_1^\infty = \frac{1}{-p+1} (\infty^{-p+1}$ ...

P: No, no! You haven't been listening! You can't just substitute x = ∞! You have to consider $\int_1^L$ and let L–>∞.

S: Fussy, fussy. Okay, I get $\int_1^L \frac{dx}{x^p} = \left[\frac{1}{-p+1} x^{-p+1} \right]_1^L = \frac{1}{p-1} - \frac{1}{p-1} L^{1-p}$ and now I let L–>∞ and get $\frac{1}{p-1}$ since L^{1-p} drops dead as L–>∞, right?

P: Even if p = 1?

S: No, I'm talking about p>1. That'd make 1-p < 0 so L^{1-p} is really L raised to a negative power and that'd make it go to zero as L–>∞. Right?

P: You've been eating smart pills again. That's very good. So what's your conclusion?

S: I conclude that ... uh ... I don't understand the question.

P: You'd conclude that, if p>1, then $y = \frac{1}{x^p}$ decrease rapidly enough for the area under the curve to be finite. That is,

$\int_1^\infty \frac{1}{x^p} dx$ converges if p > 1, to $\frac{1}{p-1}$. If p = 1, the area is infinite.

S: I knew that.

P: Guess what?

S: What.

P: We're finished the course!

S: Yeah! Yeah! Yeah!

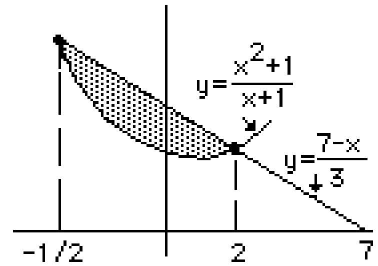
SOLUTIONS TO "ASSORTED PROBLEMS"

1. (a) $\frac{|x^2-8| - |2+x|}{x-2} = \frac{-(x^2-8) -(2+x)}{x-2} = \frac{6-x-x^2}{x-2} = \frac{(2-x)(3+x)}{x-2} = 3+x$ (for $x \neq 2$) –> 5 as x –> 2.

   (b) $\frac{f(x) - f(2)}{x - 2} = \frac{(x-2) \sin\frac{\pi}{x}}{x-2} = \sin\frac{\pi}{x} \to \sin\frac{\pi}{2} = 1$ as x –> 2

   (c) $f(t) = |\sin t| = -\sin t$ near $x = -\frac{\pi}{3}$ (since sin t < 0) hence $f'(x) = -\cos x = -\cos(-\frac{\pi}{3})$ (at $x = -\frac{\pi}{3}$) $= -\frac{1}{2}$

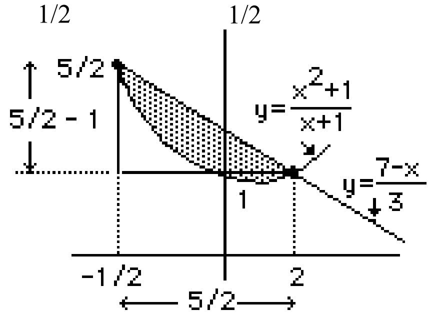
2. $y = \frac{x^2+1}{x+1} = \frac{7-x}{3}$ when $2x^2 - 3x - 2 = 0$, or $(2x+1)(x-2) = 0$ hence curves intersect when $x = \frac{-1}{2}$ and $x = 2$. For a quick sketch of $y = \frac{x^2+1}{x+1}$, note that it has a vertical asymptote at $x = -1/2$ and that $y = 1$ when $x = 0$ and $y \approx x^2/x = x$ when x is large (neglecting the constants). The area is then



$$\int_{-1/2}^{2} \left(\frac{7-x}{3} - \frac{x^2+1}{x+1} \right) dx = \left[\frac{7}{3}x - \frac{x^2}{3} \right]_{-1/2}^{2} - \int_{-1/2}^{2} \frac{x^2+1}{x+1}\, dx .$$

In this remaining integral we set $u = x+1$ so $du = \frac{du}{dx}\, dx = (1)\, dx = dx$ and change the limits: when $x = -1/2$, $u = 1/2$ and when $x = 2$, $u = 3$. We get for this integral:

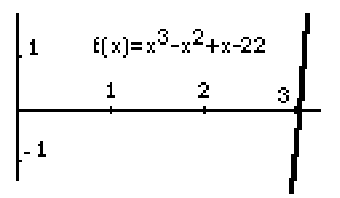
$$\int_{1/2}^{3} \frac{(u-1)^2+1}{u}\, du = \int_{1/2}^{3} \left(u - 2 + \frac{2}{u} \right) du = \left[\frac{u^2}{2} - 2u + 2\ln|u| \right]_{1/2}^{3} .$$



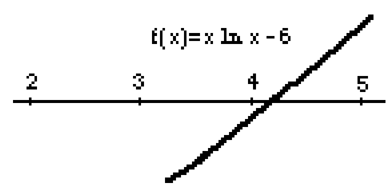
After evaluating we get an area:
$\frac{35}{6} - 2\ln 3 + 2\ln\frac{1}{2}$ or $\frac{35}{6} - 2\ln 6 \approx 2$
and we check for reasonableness by comparing with the area of the triangle shown: area of triangle = (1/2)(base)(height) = (1/2)(5/2)(5/2 - 1) = 15/8 $\approx$ 2 which seems okay.

3. $\int \frac{x+5}{x^2+4x+4}\, dx = \int \frac{x+5}{(x+2)^2}\, dx$. Let $u=x+2$, $du=dx$ and get $\int \frac{u+3}{u^2} du = \ln|u| - \frac{3}{u} + C = \ln|x+2| - \frac{3}{x+2} + C$.

4. (a) Using $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with $f(x) = x^3-x^2+x+22$ we get:



$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n + 22}{3x_n^2 - 2x_n + 1}$. The only root is slightly larger than 3, so we use $x_1=3$ and get $x_2=3.045789$, $x_3=3.044724$, $x_4=3.044723$, $x_5=3.044723$ and we conclude that the root is 3.04472 to five dec. places.

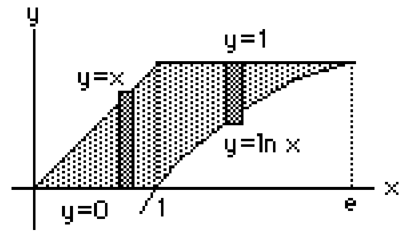


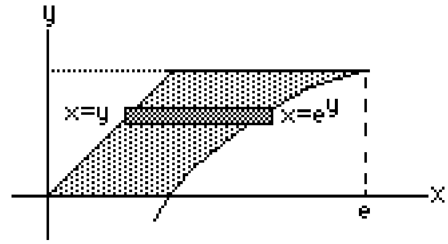
(b) With $f(x) = x \ln x - 6$ we get: $x_{n+1} = x_n - \frac{x_n \ln x_n - 6}{\ln x_n + 1}$.
There is one root between 4 and 5. If we use $x_1=4.3$ we get $x_2=4.189351$, $x_3=4.188760$, $x_4=4.188760$ and we conclude that the root is 4.18876 to five dec. places.

5. (a) $\lim_{x \to 0} \frac{1-\cos^2 x}{x^2} = \lim_{x \to 0} \frac{2\cos x \sin x}{2x} = \lim_{x \to 0} \frac{\sin 2x}{2x} = \lim_{x \to 0} \frac{2\cos 2x}{2} = \frac{2}{2} = 1$ (which is not surprising since the ratio is just $\left(\frac{\sin x}{x}\right)^2$ which has a limit of 1).

(b) $\lim_{x \to 0} \frac{e^x - 1 - x - x^2}{x^2} = \lim_{x \to 0} \frac{e^x - 1 - 2x}{2x} = \lim_{x \to 0} \frac{e^x - 2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (and this actually tells us that $e^x \approx 1 + x + \frac{x^2}{2}$ which is the quadratic approximation at $x = 0$.

6. (a)





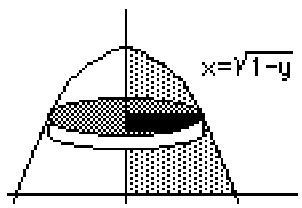
Using vertical rectangles:

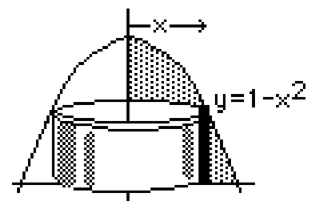
$$\text{Area} = \int_0^1 x\,dx + \int_1^e (1 - \ln x)\,dx$$

Using horizontal rectangles:

$$\text{Area} = \int_0^1 (e^y - y)\,dy$$

(b)





Using horizontal rectangles:

$$\text{Volume} = \pi \int_0^1 x^2\,dx$$

$$= \pi \int_0^1 (1-y)\,dy$$

Using vertical rectangles:

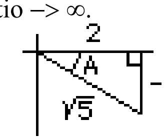
$$\text{Volume} = 2\pi \int_0^1 x\,y\,dx$$

$$= 2\pi \int_0^1 x\,(1-x^2)\,dx$$

7. (a) $\lim\limits_{x \to 0} \frac{|x+7|+|2x-3|}{x} = \lim\limits_{x \to 0} \frac{(x+7)-(2x-3)}{x}$ (since $x+7>0$ and $2x-3<0$ when $x \approx 0$) $= \lim\limits_{x \to 0} \frac{-x+10}{x}$ does not exist (since $\frac{10}{x} \to \pm\infty$ depending upon whether $x \to 0^+$ or 0^-).

(b) $\left(\sqrt{x^4+2x} - \sqrt{x^2+x}\right) \frac{\sqrt{x^4+2x} + \sqrt{x^2+x}}{\sqrt{x^4+2x} + \sqrt{x^2+x}} = \frac{x^4-x^2+x}{\sqrt{x^4+2x} + \sqrt{x^2+x}} = \frac{1 - 1/x^2 + 1/x^3}{\frac{\sqrt{x^4+2x} + \sqrt{x^2+x}}{x^4}}$ where we've divided numerator and denominator by x^4 (the highest power of x). Now $x^4 = \sqrt{x^8}$ hence the denominator becomes $\sqrt{1/x^4+2/x^7} + \sqrt{1/x^6+1/x^7} \to +0$ as $x \to -\infty$ whereas the numerator $\to 1$. Hence the ratio $\to \infty$.

8. (a) Let $A = \arctan(-\frac{1}{2})$ so that $\tan A = -\frac{1}{2}$ and A must lie in $-\pi/2 < A < \pi/2$ (the range of the arctan function). Clearly, A lies in the fourth quadrant. From the diagram, $\cos\left(\arctan(-\frac{1}{2})\right) = \cos A = \frac{2}{\sqrt{5}}$.



(b) $\frac{d}{dx}\left(\arcsin x + \frac{\sqrt{1-x^2}}{x}\right) = \frac{1}{\sqrt{1-x^2}} + \frac{x\left(\frac{-x}{\sqrt{1-x^2}}\right) - \sqrt{1-x^2}}{x^2} = \frac{1}{\sqrt{1-x^2}} + \frac{-x^2 - (1-x^2)}{x^2\sqrt{1-x^2}} = \frac{x^2 - 1}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x^2}$.

9. (a) $\frac{1}{2}\int_0^2 \frac{1}{4 + x^2}\,dx = \frac{1}{2}\left[\frac{1}{2} \arctan \frac{x}{2}\right]_0^2 = \frac{1}{4}\left(\arctan 1 - \arctan 0\right) = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$.

(b) $\displaystyle\int_{-1}^{1} 10^x\,dx = \left[\frac{1}{ln\ 10}10^x\right]_{-1}^{1} = \frac{9.9}{ln\ 10}$.

10. (a) $\displaystyle\lim_{L\to1^-}\left(\int_0^L \frac{dx}{\sqrt{1-x^2}}\right) = \lim_{L\to1^-}[\arcsin x]_0^L = \lim_{L\to1^-} \arcsin L = \arcsin 1 = \frac{\pi}{2}$.

(b) $\displaystyle\frac{d}{dx}\left(\int_{-x^2}^{x^2} t\sin t\,dt\right) = \frac{d}{dx}\left(\int_1^u t\sin t\,dt - \int^v t\sin t\,dt\right) = u\sin u\frac{du}{dx} - v\sin v\frac{dv}{dx} = 2x^3\sin x^2 - 2x^3\sin(-x^3) = 4x^3$ $\sin x^2$ where we've set $u = x^2$ and $v = -x^2$. (Note: we could have written $\displaystyle\int_{-x^2}^{x^2} t\sin t\,dt = 2\int_0^{x^2} t\sin t\,dt$ as well, since the integrand is an EVEN function of x.)

11. Write $y = f(x) = \dfrac{4x^3}{x^2+1}$, then $x = f(y) = \dfrac{4y^3}{y^2+1}$ (interchanging x and y) . Were we able to solve this latter equation for y we'd have $y = g(x)$, the inverse of $f(x)$.

(a) $f'(x) = \dfrac{(x^2+1)\,12x^2 - 4x^3(2x)}{(x^2+1)^2} = \dfrac{4x^4 + 12x^2}{(x^2+1)^2} \geq 0$ when $x \geq 0$ so f <u>has</u> an inverse there (... and everywhere else for that matter!).

(b) To find $y = g(x)$ when $x = 2$ we must solve $2 = \dfrac{4y^3}{y^2+1}$ and the solution is clearly $y = g(2) = 1$. (Note: the value $x = 2$ was chosen so you could find y by inspection!)

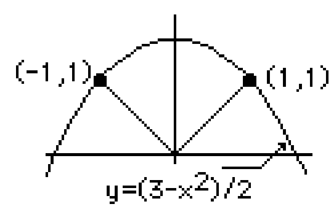
(c) To find $\dfrac{dy}{dx} = g'(x)$ when $x = 2$ (and $y = 1$) we differentiate $x = \dfrac{4y^3}{y^2+1}$ implicitly, giving

$1 = \dfrac{(y^2+1)\,12y^2 - 4y^3(2y)}{(y^2+1)^2}\dfrac{dy}{dx}$ then substitute $y = 1$ to get $1 = 4\dfrac{dy}{dx}$ so $\dfrac{dy}{dx} = g'(x) = \dfrac{1}{4}$ when $x = 2$.

(Note: Don't solve for dy/dx, <u>then</u> substitute; substitute first <u>then</u> find dy/dx ... it's much easier!)

12. We want to determine whether $e^\pi > \pi^e$ or, to put it differently, whether $e^{1/e} > \pi^{1/\pi}$ (raising both sides of the inequality to the power $\dfrac{1}{\pi e}$). Hence we consider the function $y = x^{1/x}$ to see if it's increasing or decreasing when x goes from $x = e$ to $x = \pi$. We use logarithmic differentiation: $ln\ y = \dfrac{1}{x}\ ln\ x$ so $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x\,(1/x) - ln\ x}{x^2}$ $= \dfrac{1 - ln\ x}{x^2}$ which is positive (hence $ln\ y$ is increasing) when $ln\ x < 1$, i.e. when $x < e$, and decreases thereafter. In particular, $ln\ y$, hence y, is smaller at $x = \pi$ than at $x = e$, so $\pi^{1/\pi} < e^{1/e}$ or, to put it differently, $\pi^e < e^\pi$. (In fact, $x^{1/x}$ has its maximum value at $x = e$).

13. Let (x,y) be a point on $y = (3 - x^2)/2$. Then the distance from (0,0) to (x,y) is $L = \sqrt{x^2+y^2}$ which we want to minimize. In fact, it's simpler to minimize $L^2 = x^2 + y^2$. First substitute $x^2 = 3 - 2y$ (the equation of the curve, rewritten) and get $L^2 = 3 - 2y + y^2$ so $\dfrac{d}{dy}L^2 = -2 + 2y < 0$ when $y < 1$ (so L^2 is decreasing) then $\dfrac{d}{dy}L^2 > 0$ when $y > 1$ (so L^2 is increasing). Hence the minimum occurs at $y = 1$, hence $x = \pm\sqrt{(3-y^2)/2} = \pm1$. Note that as you travel along the curve (from the third quadrant where y is negative), $\dfrac{d}{dy}L^2$ is first negative so L^2 is decreasing ... to a minimum

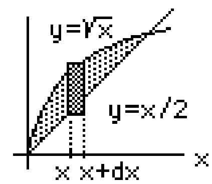


at (-1,1), then $\frac{d}{dy}$ $L^2<0$ so L^2 increases (because y > 1), then it decreases again until y = 1 again (at (1,1). (A picture is worth a thousand words!)

14. (a) $\frac{d}{dx}\left(\frac{ln\ (ln\ x)}{ln\ x}\right) = \frac{ln\ x\left(\frac{1}{ln\ x}\frac{1}{x}\right) - ln\ (ln\ x)\frac{1}{x}}{(ln\ x)^2} = \frac{1 - ln\ (ln\ x)}{x\ (ln\ x)^2}$.

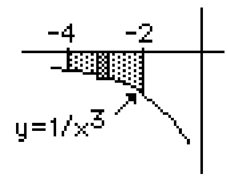
(b) $\frac{d}{dx}\left(x \cos (ln\ x)\right) = x\left(- \sin (ln\ x)\frac{1}{x}\right) + \cos (ln\ x) = \cos (ln\ x) - \sin (ln\ x)$.

15. Area = $\int_0^4 (\sqrt{x} - \frac{x}{2})\ dx$ ... using vertical rectangles



$= \left[\frac{3}{2}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \frac{3}{2}(8) - 4 = 8.$

Area = $\int_{-4}^{-2}\frac{dx}{x^3} = \left[(-\frac{1}{2}x^{-2})\right]_{-4}^{-2} = -\frac{3}{32}$, and of course we get the negative of the area because $\frac{1}{x^3}$ is negative on [-4,-2] .. so the area is actually $\frac{3}{32}$



16. $\int_{-t^2}^{t^2} \sin x^3\ dx = 0$ since the integrand is an ODD function. (Remember the test for an ODD function: f(-x) = f(x) which, for <u>our</u> f(x), becomes: $f(-x) = \sin (-x)^3 = \sin(-x^3) = - \sin x^3 = - f(x)$).

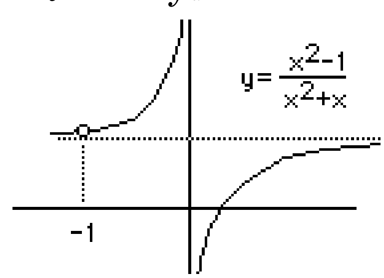
17. (a) If $y = a^x$ then $ln\ y = x\ ln\ a$ and $\frac{1}{y}\frac{dy}{dx} = ln\ a$ so $\frac{dy}{dx} = y\ ln\ a = a^x\ ln\ a$.

(b) If $y = x^x$ then $ln\ y = x\ ln\ x$ and $\frac{1}{y}\frac{dy}{dx} = 1 + ln\ x$ so $\frac{dy}{dx} = y\ (1 + ln\ x) = x^x\ (1 + ln\ x)$.

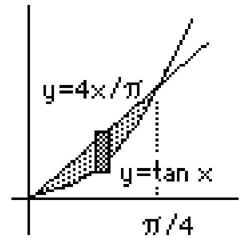
18. Let $f(x) = x^{1/3}$. Then $f(64) = 64^{1/3} = 4$ and $f'(x) = \frac{1}{3}x^{-2/3}$ so $f'(64) = \frac{1}{3(64)^{2/3}} = \frac{1}{3(4^2)} = \frac{1}{48}$. The linear approximation is: $f(x) = x^{1/3} \cong f(64) + f'(64)(x - 64) = 4 + \frac{1}{48}(x - 64)$. At x = 62 get the approximation: $62^{1/3} \approx 4 - \frac{1}{24}$

19. $\int\frac{dx}{e^x + e^{-x}}$. Let $u = e^x$ so $x = ln\ u$ and $dx = \frac{dx}{du}\ du = \frac{1}{u}\ du$. Then get $\int\frac{du/u}{u+1/u} = \int\frac{du}{u^2+1} = \arctan e^x + C$.

20. $f(x) = \frac{(x-1)(x+1)}{x(x+1)} = \frac{x-1}{x}$ provided $x \neq -1$. Hence we need only sketch $y = 1 - \frac{1}{x}$ and put an "open circle" at x = -1. Since $\lim_{x\to\infty} y = 1$, there is a horizontal asymptote: y = 1. Since $\lim_{x\to 0} y = \pm\infty$, the y-axis is a vertical asymptote. There are no critical points.



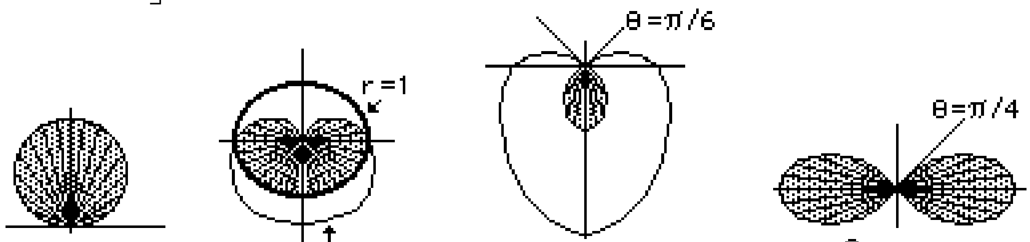
21. We sketch, and note the points of intersection: (0,0) and ($\pi/4$,1).



The area (using vertical rectangles) is $\int_0^{\pi/4} (\frac{4x}{\pi} - \tan x)\, dx$

$= \left[\frac{2x^2}{\pi} + \ln|\sec x|\right]_0^{\pi/4} = \frac{\pi}{8} + \ln\sqrt{2}$.

22.



r = sin θ r = 1 - sin θ r = 1 - 2 sin θ r^2 = cos 2θ

(a) The area inside r = sin θ is: $\frac{1}{2}\int_0^{\pi} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi} =$

$\frac{\pi}{4}$ (noting that the circle is traversed once when θ runs from 0 to π) . This agrees (of course!) with $\pi(1/2)^2$.

(b) The area inside r =1- sin θ and inside the circle r = 1 is: $2\left\{\frac{1}{2}\int_{-\pi/2}^{0} 1^2\, d\theta + \frac{1}{2}\int_0^{\pi/2} (1-\sin\theta)^2\, d\theta\right\}$ ... taking double the area to the right of the y-axis and noting that the area inside the circle is swept out when θ runs from -π/2 to 0 and the limaçon area when θ runs from 0 to π/2. This gives $\frac{\pi}{2} + \int_0^{\pi/2} (1 - 2\sin\theta + \sin^2\theta)\, d\theta$ and putting $\sin^2\theta =$

$\frac{1}{2} - \frac{\cos 2\theta}{2}$ gives $\frac{\pi}{2} + \int_0^{\pi/2} (\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta)\, d\theta$

$= \frac{\pi}{2} + \left[\frac{3\theta}{2} + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{5\pi}{4} - 2$ (about $\frac{15}{4} - 2 = 1.75$ which is somewhat larger than half the circular area: $\frac{1}{2}\pi 1^2 \approx 1.5$, so we're happy with our answer!).

(c) The area inside the smaller loop of the limaçon r = 1- 2 sin θ is: $2\left\{\frac{1}{2}\int_{\pi/6}^{\pi/2} (1-2\sin\theta)^2\, d\theta\right\}$ taking twice the area from θ = π/6 (which is the first time r goes to zero, after which r goes negative, sweeping out the inner loop) to θ = π/2 (when r is its most negative). This gives $\int_{\pi/6}^{\pi/2} (1-4\sin\theta+4\sin^2\theta)d\theta = \int_{\pi/6}^{\pi/2} (3-4\sin\theta-2\cos2\theta)d\theta$ using, once again, $\sin^2\theta = 1/2 - (1/2)\cos2\theta$. We get $[3\theta + 4\cos\theta - \sin2\theta]_{\pi/6}^{\pi/2} = \pi -$

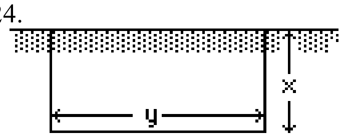
Error!.

(d) The area inside the lemniscate $r^2 = \cos 2\theta$ is: $4\left\{\frac{1}{2}\int_0^{\pi/4} \cos 2\theta\, d\theta\right\} = 4[\sin2\theta]_0^{\pi/4} = 4$.

23. If r is the radius and h the height, the surface area is $2\pi r^2 + 2\pi rh$ (the two ends plus the curved side). But r and h

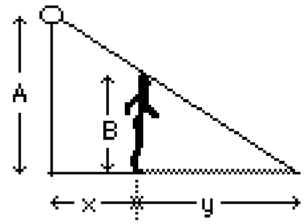
are related: volume = $\pi r^2 h = 300$ (given) so we substitute $h = 300/\pi r^2$ and get the surface area as $A(r) = 2\pi r^2+600/r$. To minimize we consider $A'(r) = 4\pi r - 600/r^2 = (4\pi r^3 - 600)/r^2$ which is first negative when r is small, (so the area is decreasing), then positive for large r (the area is increasing), so the minimum occurs for $A'(r) = 0$, or $r^3=600/4\pi$, or $r = (150/\pi)^{1/3}$ and $h = 300/\pi r^2 = 2\,(150/\pi)^{1/3}$. (Note: $h = 2r$ gives the minimum amount of aluminum no matter what the volume of the can!)

24.

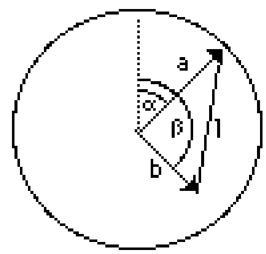


If the dimensions are x and y, then the area is xy. However, there's a relation between x and y, namely $2x + y = 1000$ (given length of fence), and if we substitute $y = 1000 - 2x$ we get the area $A(x) = x\,(1000 - 2x) = 1000x - 2x^2$. To maximize we consider $A'(x) = 1000 - 4x$ which is first positive when x is small (meaning the area is increasing), then negative when x is large (so the area is decreasing), hence the maximum area occurs when $A'(x) = 0$, or $x = 250$ meters ... hence $y = 1000 - 2x = 500$ metres.

25. We're given the rate of change $\frac{dx}{dt}$ where x gives the location of the man, and we want to know $\frac{dy}{dt}$ where y is the length of shadow. To find the relation between x and y we note the similar triangles: $\frac{A}{x+y} = \frac{B}{y}$ so that $(A-B)\,y = B\,x$. Differentiating gives $(A-B)\frac{dy}{dt} = B\frac{dx}{dt}$ and, substituting, we get $\frac{dy}{dt} = \frac{B}{A-B}\frac{dx}{dt} = \frac{2}{8-2}(3) = 1$ m/s. (Note: most "related rate" problems ask for $\frac{dy}{dt}$ when x has some value. Here, $\frac{dy}{dt} = 1$ m/s for *every* value of x.)



26. Let the minute and hour hands have lengths "a" and "b" respectively, and let their angular displacements from the 12 o'clock position be α and β respectively. (I like symbols better than numbers!) The distance between them is given by the "cosine law" for triangles: $l^2 = a^2 + b^2 - 2ab\cos(\beta - \alpha)$. We know that $\frac{d\beta}{dt} = \frac{2\pi \text{ radians}}{12 \text{ hours}} = \frac{\pi}{6}$ radians/hour and $\frac{d\alpha}{dt} = \frac{2\pi \text{ radians}}{1 \text{ hour}} = 2\pi$ radians/hour and we wish to know $\frac{dl}{dt}$ when $\alpha = 0$ and $\beta = \frac{\pi}{2}$ (i.e 3 o'clock) . Differentiating the relation between l, α and β gives:

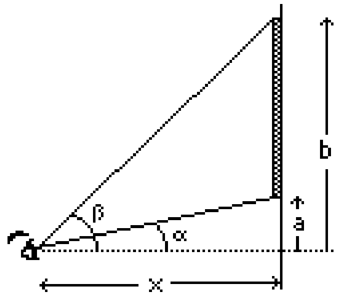


$2l\frac{dl}{dt} = 2ab\sin(\beta - \alpha)\left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right)$ and substituting what we're given (including $l = \sqrt{3^2+4^2} = 5$, at 3 o'clock) gives $\frac{dl}{dt} = \frac{(3)(4)}{5}\sin\frac{\pi}{2}\left(\frac{\pi}{6} - 2\pi\right) = -\frac{61\pi}{15}$ cm/hour. (A negative sign, so the tips are getting closer together.)

27. For $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$ and $f'''(x) = \sin x$. The "familiar" angle nearest to 57˚ is 60˚, or $x = \pi/3$, so we get the cubic approximation: $f(\frac{\pi}{3}) + f'(\frac{\pi}{3})(x-\frac{\pi}{3}) + f''(\frac{\pi}{3})\frac{(x-\frac{\pi}{3})^2}{2!} + f'''(\frac{\pi}{3})\frac{(x-\frac{\pi}{3})^3}{3!}$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{2}\frac{(x - \frac{\pi}{3})^2}{2!} + \frac{\sqrt{3}}{2}\frac{(x - \frac{\pi}{3})^3}{3!}$ and, for $x - \frac{\pi}{3} = -3\frac{\pi}{180} = -\frac{\pi}{60}$ (which is - 3˚, expressed in RADIANS!) , we get $\cos 57^\circ \approx .54463888$ (compared to the "exact" value of .54463903).

28.



Let the top of the picture be "b" metres above your eye, the bottom "a" metres. If you stand a distance "x" metres from the wall, the angle subtended by the picture at your eye is β - α (see diagram) where $\tan \beta = \frac{b}{x}$ (so $\beta = \arctan \frac{b}{x}$) and $\tan \alpha = \frac{a}{x}$ (so $\alpha = \arctan \frac{a}{x}$). We maximize $\beta - \alpha = f(x) = \arctan \frac{b}{x} - \arctan \frac{a}{x}$ for $x > 0$. First, we compute $f'(x) = \frac{1}{1+(b/x)^2}\left(\frac{-b}{x^2}\right) - \frac{1}{1+(a/x)^2}\left(\frac{-a}{x^2}\right) = \frac{(b-a)(ab - x^2)}{(x^2+a^2)(x^2+b^2)}$ which is first positive (when x is small) so that the angle is increasing, then $f'(x) < 0$ when x is large ... so the maximum occurs when $f'(x) = 0$, at $x = \sqrt{ab}$ (which has the correct dimensions; if a and b are in metres, so is x) . For b = 3 and a = 1 we get $x = \sqrt{3}$ metres. (Note: although we're given the dimensions "a" and "b", I choose to give them labels so I can check the dimensions ... and am less likely to make errors in arithmetic: I can always multiply "a" by "b" and get "ab" ... I'm not so sure about 3 and 1.)

29. To find a point of intersection of $xy = \sqrt{2}$ and $x^2 - y^2 = 1$, put $y = \frac{\sqrt{2}}{x}$ into the second equation, giving: $x^2 - \frac{2}{x^2} = 1$, or $x^4 - x^2 - 2 = 0$, or $(x^2 - 2)(x^2+1) = 0$, hence $x^2 - 2 = 0$ and $x = \pm\sqrt{2}$, so $y = \frac{\sqrt{2}}{x} = \pm 1$. There are two points of intersection: $(\sqrt{2}\ ,1)$ and $(-\sqrt{2}\ ,-1)$. At these points we expect the tangent lines to be perpendicular: $\frac{d}{dx}(xy = \sqrt{2})$ gives $\frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{\sqrt{2}}$ for the first curve (at *both* points of intersection) and $\frac{d}{dx}(x^2-y^2=1)$ gives $\frac{dy}{dx} = \frac{x}{y} = \sqrt{2}$ (at *both* points of intersection) . The product of these slopes is -1, hence they are perpendicular.

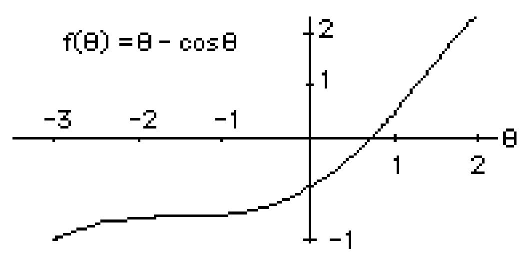
30. (a) $\frac{d}{dx}(\sqrt{x} + \sqrt{y} = 1)$ gives $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$. At the point (a,b), the tangent has slope $-\sqrt{\frac{b}{a}}$ and the tangent line is: $\frac{y - b}{x - a} = -\sqrt{\frac{b}{a}}$ or $y - b = -\sqrt{\frac{b}{a}}\,(x - a)$ or $y = b - \sqrt{\frac{b}{a}}\,(x - a)$.

(b) Put x = 0 to get the y-intercept, namely $y = b + a\sqrt{\frac{b}{a}} = b + \sqrt{ab}$.

Put y = 0 to get the x-intercept, namely $x = a + b\sqrt{\frac{a}{b}} = a + \sqrt{ab}$.

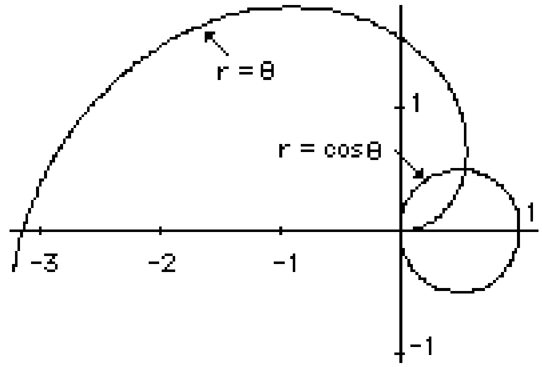
(c) The sum of the intercepts is $a + 2\sqrt{ab} + b = (\sqrt{a} + \sqrt{b})^2 = 1^2 = 1$.
(Note: $\sqrt{a} + \sqrt{b} = 1$, since (a,b) lies on the curve.)

31. Curves r=θ, r=cos θ intersect where f(θ)=θ-cos θ=0. A plot of f(θ) indicates a single root near θ=1, so we start Newton's method with θ_1=1.



The Newton iterations satisfy:

$\theta_{n+1} = \theta_n - \dfrac{f(\theta_n)}{f'(\theta_n)} = \theta_n - \dfrac{\theta_n - \cos\theta_n}{1+\sin\theta_n}$. Starting with θ_1=1 we get: $\theta_2 = 0.75036387$, $\theta_3 = 0.73911289$, $\theta_4 = 0.73908513$, $\theta_5 = 0.73908513$ and conclude that the point of intersection occurs at $\theta = 0.73908513$ radians (about 42 degrees) so $r = \theta = 0.73908513$ as well.



32. Consider $f(x) = \pi(1 - x) - 2x\, ln \frac{1}{x} = \pi(1 - x) + 2x\, ln\, x$ (noting that $ln \frac{1}{x} = - ln\, x$). To show that $f(x) > 0$, we'll find the minimum value of f(x) and hope that it's positive. First we investigate where f(x) is increasing or decreasing using $f'(x) = -\pi + 2(1 + ln\, x)$ which is negative at first (when x is near 0 since $ln\, x \rightarrow -\infty$ when $x \rightarrow 0^+$) then positive when x is large. The minimum then occurs when $-\pi + 2(1 + ln\, x) = 0$ or $ln\, x = \frac{\pi}{2} - 1$ or $x = e^{\pi/2 - 1}$. Substituting into $\pi(1 - x) + 2x\, ln\, x$ gives $\pi - 2\, e^{\pi/2 - 1}$ which is positive.

33. (a) $\displaystyle\int_0^L \frac{dx}{1+x^2} = [\arctan x]_0^L = \arctan L - \arctan 0 = \arctan L \rightarrow \frac{\pi}{2}$ as $L \rightarrow \infty$.

(b) In $\displaystyle\int_1^L \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$, let $u = \sqrt{x}$ so $du = \frac{du}{dx}\, dx = \frac{dx}{2\sqrt{x}}$ and as x goes from 1 to L, u goes from $\sqrt{1}$ to $\sqrt{L}$. We get $\displaystyle\int_1^{\sqrt{L}} 2 \sin u\, du = [-2 \cos u]_1^{\sqrt{L}} = 2 \cos (1) - 2 \cos\sqrt{L}$ which has no limit as $L \rightarrow \infty$, hence $\displaystyle\int_1^{\infty} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ doesn't exist (i.e this improper integral diverges).

(c) $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\, dx = \int_{-\infty}^{0} e^{x}\, dx + \int_0^{\infty} e^{-x}\, dx$ (where we've split the integral and set $|x| = -x$ when $x \le 0$ and $|x| = x$ when $x \ge 0$) . We have that

$\displaystyle\int_{-\infty}^{0} e^{x}\, dx = \lim_{A \rightarrow -\infty} \int_A^0 e^{x}\, dx = \lim_{A \rightarrow -\infty} [e^{x}]_A^0 = \lim_{A \rightarrow -\infty} (1 - e^{A}) = 1$ and, in a similar manner,

$\displaystyle\int_0^{\infty} e^{-x}\, dx = \lim_{B \rightarrow \infty} \int_0^B e^{-x}\, dx = \lim_{B \rightarrow \infty} [- e^{-x}]_0^B = \lim_{B \rightarrow \infty} (1 - e^{-B}) = 1$. Hence $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\, dx = 1 + 1 = 2$.

(d) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$ and $\displaystyle\int_0^1 \frac{dx}{x^2} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x^2} = \lim_{a \rightarrow 0^+} \left[\frac{-1}{x}\right]_a^1 = \lim_{a \rightarrow 0^+} \left(\frac{1}{a} - 1\right) = \infty$ so $\displaystyle\int_0^1 \frac{dx}{x^2}$ diverges hence so does $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$

(e) $\displaystyle\int_{L}^{\pi} \frac{dx}{x^{1/3}} = \left[\frac{3}{2} x^{2/3}\right]_L^{\pi} = \frac{3}{2}\pi^{3/2} - \frac{3}{2} L^{3/2} \to \frac{3}{2}\pi^{3/2}$ as $L \to 0$, so $\displaystyle\int_{0}^{\pi} \frac{dx}{x^{1/3}}$ converges to $\frac{3}{2}\pi^{3/2}$.

(f) $\displaystyle\int_{-1}^{L} \frac{dx}{x-1} = [ln\,|x - 1|]_{-1}^{L} = ln\,|L - 1| - ln\,2 \to -\infty$ as $L \to 1^-$ (recall that $ln\,|L - 1| \to -\infty$ as $|L - 1| \to 0^+$). Hence,

$\displaystyle\int_{-1}^{1} \frac{dx}{x-1}$ has no value ... or "diverges" ... or "doesn't exist".

(g) For $\int ln\,x\,dx$ let $u = ln\,x$ and $dv = dx$ so $du = \frac{du}{dx}\,dx = \frac{1}{x}\,dx$ and $v = x$ and we get (integrating by parts)

$\int u\,dv = uv - \int v\,du = x\,ln\,x - \int x\frac{1}{x}\,dx = x\,ln\,x - x + C$. Hence:

$\displaystyle\int_{L}^{1} ln\,x\,dx = [x\,ln\,x - x]_L^1 = \{(1)ln\,(1) - 1\} - \{L\,ln\,L - L\} = -1 + L - L\,ln\,L$ (noting that $ln\,1 = 0$)

and now we must evaluate $\displaystyle\lim_{L\to 0^+} L\,ln\,L$. If we rewrite $L\,ln\,L$ as $\dfrac{ln\,L}{\frac{1}{L}}$ it then has the form $\frac{\infty}{\infty}$ and we can use

l'Hopital's rule: $\displaystyle\lim_{L\to 0^+} \frac{ln\,L}{\frac{1}{L}} = \lim_{L\to 0^+}\left(\frac{\frac{1}{L}}{-\frac{1}{L^2}}\right) = \lim_{L\to 0^+} (-L) = 0$. Hence $\displaystyle\int_{0}^{1} ln\,x\,dx = \lim_{L\to 0^+}\int_{L}^{1} ln\,x\,dx = -1$.

(The fact that's it's negative should be no surprise: $ln\,x < 0$ when $0 < x < 1$).